CHAPTER 5

A

Central Angles, Arc Length, and Sector Area

You will need
- a calculator
- a compass
- a protractor

► **GOAL**

Identify central angles and determine arc length and sector area formed by a central angle.

Learn about the Math

An angle whose vertex is the centre of a circle and whose sides pass through a pair of points on the circle is called a **central angle**. The following symbol, pronounced "theta", is used to represent a central angle: θ.

Each central angle forms an **arc length** between the points that it passes through on the circle.

The arc length of a circle can be found by first calculating what fraction of the circle is represented by the central angle. Then you find the circumference of the circle. Finally, you find the fraction of that circumference.

For example, if a circle has a radius of 10.0 cm and a central angle of 60°, to calculate the arc length for that angle we must first determine what fraction of the circle is represented by the central angle 60°. Since there are 360° in a full circle, we can find the fraction of a circle by simplifying $\frac{60}{360}$; $\frac{60}{360} = \frac{1}{6}$. The sector represents $\frac{1}{6}$ of the circle.

To find the arc length, we now need to find the circumference of the circle. Circumference of a circle is equal to $2\pi r$.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 10.0 \\ &\doteq 62.8 \text{ cm} \end{aligned}$$

The circumference of this circle is about 62.8 cm. To find the arc length, we need to find $\frac{1}{6}$ of 62.8 cm.

$$\frac{1}{6} \times 62.8 \doteq 10.5 \text{ cm}$$

The arc length formed by this central angle is about 10.5 cm.

central angle
an angle whose vertex is the centre of a circle and whose sides pass through a pair of points on the circle

arc length
the length of a section of a circle's circumference

To make the process quicker, you can refer to the formula used to determine the arc length formed by a central angle:

$C_A = \frac{\theta}{360°} \times 2\pi r$, where C_A is the arc length, and r is the radius of the circle.

A **sector of a circle** is a pie-shaped region bounded by an arc and an angle. If you know the radius of a circle and a central angle, you can determine the sector area for that angle by first calculating the fraction of the circle that the sector represents, then finding the area of the entire circle, and finally calculating that fraction of the total area.

sector of a circle
pie-shaped region bounded by an arc and an angle

For example, if we again use a circle with a radius of 10 cm and a central angle of 60°, we can determine the sector area by first finding the fraction of the circle, which we calculated earlier to be $\frac{1}{6}$ and then calculating the area of the entire circle. The area of a circle is equal to πr^2. The area of this circle can be found using this formula.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi (10)^2 \\ &\doteq 314 \text{ cm}^2 \end{aligned}$$

Now we need to find $\frac{1}{6}$ of 314 cm^2 to determine the area of the sector formed by this central angle.

$$\frac{1}{6} \times 314 \doteq 52.3 \text{ cm}^2$$

The area of the sector formed by central angle is about 52.3 cm^2.

The formula used to determine the sector area for any central angle is $A_S = \frac{\theta}{360°} \times \pi r^2$, where A_S is the area of the sector and r is the radius of the circle.

Denise needs to determine the arc length and sector area formed by a 40° central angle on a circle with a radius of 5 cm.

? How can Denise calculate the arc length and sector area for this central angle?

A. Sketch a circle on a sheet of paper with a radius of 5 cm.

B. Use a protractor to draw a 40° central angle on the circle.

C. Find the arc length between the two points where this angle passes through the circle, using the formula

$C_A = \frac{\theta}{360°} \times 2\pi r.$

D. Find the area for the sector of this circle formed by the central angle and the arc formed in step B, using the formula $A_s = \frac{\theta}{360°} \times \pi r^2$.

Reflecting

1. What is a central angle of a circle?
2. What is the formula for finding the arc length formed by the points where a central angle passes through the circle?
3. Why must you calculate the area of the entire circle in order to calculate the area of just one sector?

Work with the Math

Example 1: Determining the arc length and sector area for a central angle of a circle

Calculate the arc length and the area of a sector formed by a 30° central angle on a circle with a radius of 2 cm.

Li Ming's Solution

First, I will calculate the arc length. To do this I must first determine what fraction of the circle is represented by the central angle 30°. Since there are 360° in a full circle, I can find the fraction of a circle by simplifying $\frac{30}{360}$; $\frac{30}{360} = \frac{1}{12}$. The sector represents $\frac{1}{12}$ of the circle.

To find the arc length, I now need to find the circumference of the entire circle. Circumference of a circle is equal to $2\pi r$.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 2 \\ &= 2 \times 3.14 \times 2 \\ &\doteq 12.56 \text{ cm} \end{aligned}$$

The circumference of this circle is about 12.56 cm. To find the arc length, I need to find $\frac{1}{12}$ of 12.56 cm.

$$\frac{1}{12} \times 12.56 \doteq 1.047 \text{ cm}$$

The arc length formed by this central angle is about 1.047 cm.

Next, I will determine the area of the sector by calculating the area of the entire circle and then multiplying by $\frac{1}{12}$. The area of a circle is equal to πr^2. The area of this circle then can be found using this formula.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi (2)^2 \\ &\doteq 12.56 \text{ cm}^2 \end{aligned}$$

Now I need to find $\frac{1}{12}$ of 12.56 cm^2, to determine the area of the sector formed by this central angle.

$$\frac{1}{12} \times 12.56 \doteq 1.047 \text{ cm}^2$$

The area of the sector formed by this central angle is about 1 cm^2.

A Checking

4. Calculate the arc length and the area of the sector formed by a 20° central angle on a circle with a radius of 15 cm.

B Practising

5. Complete each of the following statements with a term that will make it a true statement.

a) A __________ __________ of a circle is an angle whose vertex is the centre of a circle and whose sides pass through a pair of points on the circle.

b) A __________ of a circle is a pie-shaped region bounded by an arc and an angle.

c) __________ __________ is the length of a section of a circle's circumference.

6. Determine the area of the sector formed by each of the following central angles, on a circle with the given radius or diameter.

a) central angle: 40°
radius: 5 m

b) central angle: 55°
diameter: 28 cm

c) central angle: 60°
radius: 15.0 cm

d) central angle: 90°
diameter: 16.0 cm

7. Determine the arc length formed by each of the following central angles, on a circle with the given radius or diameter.

   a) central angle: 25°
      diameter: 22 m
   b) central angle: 10°
      radius: 25 cm
   c) central angle: 15°
      diameter: 66 mm
   d) central angle: 80°
      radius: 6.0 m

8. Determine the arc length and area for the sectors formed by each of the following central angles, on a circle with the given radius or diameter.

   a) central angle: 70°
      radius: 4 m
   b) central angle: 75°
      diameter: 20 cm
   c) central angle: 40°
      radius: 3 m
   d) central angle: 105°
      diameter: 18 cm

C Extending

9. For a particular circle, the central angle is $x°$. The corresponding arc length is C_A, and the corresponding sector area is A_S. For this circle, what would be the arc length and sector area corresponding to a central angle of $2x°$?

10. For a particular circle with radius r, the arc length corresponding to a central angle of $x°$ is C_A and the corresponding sector area is A_S. For a circle with radius $2r$ what would be the arc length and sector area corresponding to a central angle of $x°$?