# 1 INDUCTIVE AND DEDUCTIVE REASONING

## Specific Outcomes Addressed in the Chapter

### WNCP

**Logical Reasoning**
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. \([C, CN, PS, R]\) \([1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7]\)
2. Analyze puzzles and games that involve spatial reasoning, using problem solving strategies. \([CN, PS, R, V]\) \([1.7]\)

## Achievement Indicators Addressed in the Chapter

### Logical Reasoning

1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning. \([1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7]\)
1.2 Explain why inductive reasoning may lead to a false conjecture. \([1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7]\)
1.3 Compare, using examples, inductive and deductive reasoning. \([1.4, 1.6, 1.7]\)
1.4 Provide and explain a counterexample to disprove a given conjecture. \([1.3, 1.4, 1.5, 1.6, 1.7]\)
1.5 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies or algebraic number tricks. \([1.4]\)
1.6 Prove a conjecture, using deductive reasoning (not limited to two column proofs). \([1.4]\)
1.7 Determine if a given argument is valid, and justify the reasoning. \([1.2, 1.4, 1.5, 1.6, 1.7]\)
1.8 Identify errors in a given proof; e.g., a proof that ends with \(2 = 1\). \([1.5]\)
1.9 Solve a contextual problem that involves inductive or deductive reasoning. \([1.4, 1.6, 1.7]\)
2.1 Determine, explain and verify a strategy to solve a puzzle or to win a game. \([1.7]\)
2.2 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game. \([1.7]\)
2.3 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game. \([1.7]\)

## Prerequisite Skills Needed for the Chapter

This chapter, while focusing on new learning related to inductive and deductive reasoning, provides an opportunity for students to review the following skills and concepts:

### Shape and Space
- Determine parallel side lengths in parallelograms and other quadrilaterals.
- Draw diagonals in rectangles and medians in triangles.
- Identify vertically opposite angles and supplementary angles in intersecting lines.

### Patterns and Relations
- Represent a situation algebraically.
- Simplify, expand, and evaluate algebraic expressions.
- Solve algebraic equations.
- Factor algebraic expressions, including a difference of squares.
- Apply and interpret algebraic reasoning and proofs.
- Interpret Venn diagrams.

### Number
- Identify powers of 2, consecutive perfect squares, prime numbers, and multiples.
- Determine square roots and squares.

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Chapter 1 Introduction
# Chapter 1: Planning Chart

<table>
<thead>
<tr>
<th>Lesson (SB)</th>
<th>Charts (TR)</th>
<th>Pacing (14 days)</th>
<th>Key Question/Curriculum</th>
<th>Materials/Masters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Getting Started, pp. 4–5</td>
<td>Planning, p. 4 Assessment, p. 6</td>
<td>2 days</td>
<td></td>
<td>Review of Terms and Connections, Diagnostic Test</td>
</tr>
<tr>
<td>1.1: Making Conjectures: Inductive Reasoning, pp. 6–15</td>
<td>Planning, p. 7 Assessment, p. 12</td>
<td>1 day</td>
<td>Q9 LR1 [C, CN, PS, R]</td>
<td>calculator, compass, protractor, and ruler, or dynamic geometry software, tracing paper (optional)</td>
</tr>
<tr>
<td>1.2: Exploring the Validity of Conjectures, pp. 16–17</td>
<td>Planning, p. 14 Assessment, p. 16</td>
<td>1 day</td>
<td>LR1 [CN, PS, R]</td>
<td>Explore the Math: Optical Illusions, ruler, calculator</td>
</tr>
<tr>
<td>1.3: Using Reasoning to Find a Counterexample to a Conjecture, pp. 18–25</td>
<td>Planning, p. 17 Assessment, p. 20</td>
<td>1 day</td>
<td>Q14 LR1 [C, CN, R]</td>
<td>calculator, ruler, compass</td>
</tr>
<tr>
<td>1.4: Proving Conjectures: Deductive Reasoning, pp. 27–33</td>
<td>Planning, p. 24 Assessment, p. 28</td>
<td>1 day</td>
<td>Q10 LR1 [PS, R]</td>
<td>calculator, ruler</td>
</tr>
<tr>
<td>1.5: Proofs That Are Not Valid, pp. 36–44</td>
<td>Planning, p. 30 Assessment, p. 33</td>
<td>1 day</td>
<td>Q7 LR1 [C, CN, PS, R]</td>
<td>grid paper, ruler, scissors</td>
</tr>
<tr>
<td>1.6: Reasoning to Solve Problems, pp. 45–51</td>
<td>Planning, p. 35 Assessment, p. 38</td>
<td>1 day</td>
<td>Q10 LR1 [C, CN, PS, R]</td>
<td>calculator</td>
</tr>
<tr>
<td>1.7: Analyzing Puzzles and Games, pp. 52–57</td>
<td>Planning, p. 39 Assessment, p. 42</td>
<td>1 day</td>
<td>Q7 LR2 [CN, PS, R]</td>
<td>counters in two colours or coins of two denominations, toothpicks (optional), paper clips (optional), Solving Puzzles (Questions 10 to 13)</td>
</tr>
</tbody>
</table>
1 OPENER

Using the Chapter Opener

Discuss the photograph, and hypothesize about what happened in the previous half hour. You could set up a role-playing situation, in which groups of four students could take the roles of driver 1, driver 2, an eyewitness, and an investigator. Together, the four students could develop questions and responses that would demonstrate their conjectures about what led up to the events seen in the photograph. This could be set up as a series of successive interviews between the investigator and the other three people in the situation.

Tell students that, in this chapter, they will be examining situations, information, problems, puzzles, and games to develop their reasoning skills. They will form conjectures through the use of inductive reasoning and prove their conjectures through the use of deductive reasoning.

In Math in Action on page 15 of the Student Book, students will have an opportunity to revisit an investigative scenario through conjectures, witness statements, and a diagram. You may want to discuss the links among reasoning, evidence, and proof at that point.
1 GETTING STARTED

The Mystery of the Mary Celeste

Introduce the activity by showing a map of the area from New York to the Bay of Gibraltar. Have students work in pairs. Ask them to imagine the challenges of travelling this distance by water in the present time. How would the challenges have been different in 1872? Discuss these challenges as a class, and then ask students to read the entire activity before responding to the prompts.

After students finish, ask them to share their explanations and justifications. Discuss whether one explanation is more plausible than another.

Sample Answers to Prompts

A. Answers may vary, e.g., there were four significant pieces of evidence:
   1. The hull was not damaged.
   2. No boats were on board.
   3. Only one pump was working.
   4. The navigation instruments, ship’s register, and ship’s papers were gone.

B. Answers may vary, e.g., the bad weather could have scared the crew into thinking that the alcohol they were carrying was going to catch fire. The captain and crew might have opened the hatches and then got into the lifeboats to be safe.

C. Answers may vary, e.g., the bad weather could have been severe enough to cause water to be washing over the bow of the ship. Since only one pump was working, perhaps the water level was rising inside the ship. If the crew could not pump all the water out of the ship, they might have opened the hatches at the front and the back to help bail out the water. If the water continued to rise, the captain and crew might have taken the navigational equipment and the ship’s register and papers, and abandoned ship into the lifeboats. If they left the ship during bad weather, they might have lost contact with the Mary Celeste and their lifeboats might have sunk.

D., E., F. Answers may vary, e.g., a piece of evidence that would support the explanation would be confirmation that lifeboats had been aboard when the Mary Celeste left New York Harbour.
Background

- This mystery is true and well documented in court records. Charles Edey Fay's book (published in 1942 and reprinted in 1988) about the mystery is a factual study of the case, unlike Arthur Conan Doyle's short story (published in 1884), which blends facts of the case with many pieces of fiction. Conan Doyle used the basic facts in the historical records but took liberties by suggesting that the crew of the Mary Celeste had departed only a very short time before the crew of the Dei Gratia spotted the ship. Suggestions of tea still steaming in cups and items still fresh in the galley (ship's kitchen) could not have been true, based upon the first-hand data entered into factual evidence.

- In August 2001, the wreck of the Mary Celeste was located off the coast of Haiti. The key words "Mary Celeste" and "mystery" entered into an Internet search engine will yield more information about the mystery. As well, books have been written about the mystery, but some ascribe details that are not supported by the evidence in the historical accounts.

What Do You Think? page 5

Use this activity to activate knowledge and understanding about inductive and deductive reasoning. Explain to students that the statements involve math concepts or skills they will learn in the chapter—they are not expected to know the answers now. Ask students to read each statement, think about it, and decide whether they agree or disagree with it. Have volunteers explain the reasons for their decisions. Students can share their reasoning in small groups, in groups where all agree or disagree, or in a general class discussion. Tell students that they will revisit their decisions at the end of the chapter.

Sample Answers to What Do You Think?

The correct answers are indicated with an asterisk (*). Students should be able to give the correct answers by the end of the chapter.

1. Agree. Answers may vary, e.g., patterns can be represented by expressions that show how the patterns change.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>●</th>
<th>●●</th>
<th>●●●</th>
<th>●●●●</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure Number (f)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Number of Dots</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

The pattern is represented by the expression $2f$.

*Disagree. Answers may vary, e.g., a pattern over a short time may not be true all the time. Seeing four people exit a shop with coffee cups in their hands does not mean that the next person leaving the shop will be holding a coffee cup.

2. *Agree. Answers may vary, e.g., a pattern may be seen after examining several examples. After seeing four people exiting a shop with coffee cups, a prediction can be made that the shop sells coffee. However, more evidence is needed.
Disagree. Answers may vary, e.g., a pattern shows only what was and not what will be. More evidence is needed to make a reliable prediction.

3. Agree. Answers may vary, e.g., the pattern shows increasing squares of numbers: $1^2, 2^2, 3^2, 4^2, 5^2$, so the next three terms are $6^2, 7^2$, and $8^2$.
   *Disagree. Answers may vary, e.g., the pattern can be described as increasing squares, but it can also be described as the sum of the preceding number and the next odd number: $0 + 1, 1 + 3, 4 + 5, 9 + 7, 16 + 9$. In both descriptions of the pattern, however, the next three terms would be 36, 49, and 64.

### Initial Assessment for Learning

<table>
<thead>
<tr>
<th>What You Will See Students Doing...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students decide that some pieces of evidence are more important than others.</td>
<td>Students place equal value on all pieces of evidence.</td>
</tr>
<tr>
<td>Students make inferences about the patterns that the evidence presents.</td>
<td>Students make predictions that do not take into account the evidence available.</td>
</tr>
<tr>
<td>Students justify their predictions based on the evidence available.</td>
<td>Students are unable to develop a justification that is clear and reasonable.</td>
</tr>
</tbody>
</table>

### Differentiating Instruction | How You Can Respond

**EXTRA SUPPORT**

1. If students have difficulty identifying the most important pieces of evidence, scaffold the task by examining the pieces of evidence in sets of three. Ask: Of these three pieces of evidence, which is the most important? Limiting the range of possibilities makes choices easier to make.

2. If students have difficulty visualizing the state of the ship when found by the crew of the *Dei Gratia*, then accessing blueprints for a ship of that type and size may be helpful. Students can do a search using key words such as “ship’s plans” and “boat building” to look for these blueprints.

Use *Review of Terms and Connections*, Teacher’s Resource pages 53 to 56, to activate students’ skills.
1.1 MAKING CONJECTURES: INDUCTIVE REASONING

Lesson at a Glance

Prerequisite Skills/Concepts
- Identify perfect squares, prime numbers and odd and even integers.
- Determine parallel side lengths in parallelograms and other quadrilaterals.
- Draw diagonals in rectangles and medians in triangles.

WNCP
Specific Outcome
Logical Reasoning
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators
1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.

Math Background
- This lesson provides an opportunity for students to develop their understanding of inductive reasoning through the mathematical processes of communication, connections, problem solving, and reasoning.
- Communication is promoted by sharing conjectures, while connections are made using the contexts presented, the evidence given, and the conjectures developed. Both communication and connections become integral parts of reasoning, as students justify the conjectures they have developed based on the context and evidence.
- Problem solving is established through the variety of interpretations possible, based on the given context and evidence. This, in turn, promotes communication about the different interpretations and justifications.

Student Book Pages 6–15

GOAL
Use reasoning to make predictions.

Preparation and Planning

Pacing
10 min Introduction
35–45 min Teaching and Learning
10–15 min Consolidation

Materials
- calculator
- compass, protractor, and ruler, or dynamic geometry software
- tracing paper (optional)

Recommended Practice
Questions 3, 4, 6, 10, 14, 16

Key Question
Question 9

New Vocabulary/Symbols
conjecture
inductive reasoning

Mathematical Processes
- Communication
- Connections
- Problem Solving
- Reasoning

Nelson Website
http://www.nelson.com/math

1.1: Making Conjectures: Inductive Reasoning
1 Introducing the Lesson

(10 min)

Explore (Pairs, Class), page 6
The Explore problem can be assigned for students to discuss in pairs, or it can be discussed as a class. It provides an opportunity for students to make a conjecture based on given evidence and to develop justification for their conjecture. The following questions may help students:

- Where might you have seen this sequence?
- How could this sequence be part of a pattern?

Have students share their explanations with the class. Encourage different conjectures for the given sequence, and explore the possibility that more than one conjecture may be correct.

Sample Solutions to Explore

- If the colour sequence is red, orange, and yellow, the rest of the sequence may be green, blue, and purple. These colours are the primary and secondary colours seen on a colour wheel.
- If the colour sequence is red, orange, and yellow, the rest of the sequence may be green, blue, indigo, and violet. These colours are those of a rainbow.
- If the colour sequence is red, orange, and yellow, the rest of the sequence may repeat these three colours.

2 Teaching and Learning

(35 to 40 min)

Investigate the Math (Class), page 6
This investigation allows students to discuss patterning and the prediction about the 10th figure. Help students understand that the pattern focuses on the congruent unit triangles, not on the different-sized triangles.

Math Background

- To make conjectures that are valid, based on a pattern of evidence, students need to have a variety of sample cases to view. Since any pattern requires multiple cases to support it, more than one or two specific cases are needed to begin to formulate a conjecture. The more cases that fit the conjecture, the stronger the validity of the conjecture becomes. The strength of a conjecture, however, does not substitute for proof. Proof comes only when all cases have been considered.
Sample Answers to Prompts

A. | Figure | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
   | Number of Triangles | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |

B. The pattern in the table shows that the number of triangles equals the square of the figure number.

C. [Figure 4 and Figure 5]

D. Figure 11 has $11^2$ or 121 triangles. Figure 12 has $12^2$ or 144 triangles.

The numeric pattern in the table shows that each figure will have a perfect square of congruent triangles. The number of congruent triangles in each figure is the square of the figure number.

Reflecting, page 6

Students can work on the Reflecting questions in pairs, before or instead of a class discussion.

Sample Answers to Reflecting

E. Georgia’s conjecture is reasonable because, when the table is extended to the 10th figure, the pattern of values is the same as Georgia’s prediction.

F. Georgia used inductive reasoning by gathering evidence about more cases. This evidence established a pattern. Based on this pattern, Georgia made a prediction about what the values would be for a figure not shown in the evidence.

G. A different conjecture could be made because a different pattern could have been seen. If the focus had been only on the congruent triangles with their vertices at the bottom and their horizontal sides at the top, then the following conjecture could have been made: The 5th figure will have 10 congruent triangles.
Consolidation
(10 to 15 min)

Apply the Math (Class, Pairs), pages 7 to 11
Using the Solved Examples

In Example 1, a conjecture is developed based on the evidence for annual rainfall. Students should be encouraged to explain, in their own words, how and why Lila came up with her conjecture. When discussing the example, focus on the patterns that have been identified. Encourage students to explain whether the reasoning makes sense.

In Example 2, a conjecture about the product of odd integers is developed. Students are encouraged to discuss the limited number of examples that Jay used to make his conjecture. Does the quantity of evidence make the conjecture more or less believable? What other evidence might Jay have used? How does the evidence that Jay did use show a pattern?

Example 3 presents two different methods for developing a conjecture about the difference between consecutive perfect squares: numerically and geometrically. Students are encouraged to discuss the strengths of both conjectures and the evidence on which each was developed.

In Example 4, two different methods are used to develop conjectures about the shape created by joining the midpoints of adjacent sides in a quadrilateral: using a protractor and ruler or using dynamic geometry software. Encourage students to test Marc’s and Tracey’s solutions to reinforce geometric understanding and construction skills. Sorting quadrilaterals in a Venn diagram to look for common and unique attributes of different quadrilaterals could be a reminder activity prior to studying Example 4. Ask the following questions to guide students through the solutions:

• How did Marc decide to focus upon a parallelogram? What pattern did Marc recognize before he made his conjecture? How did Marc’s use of three different ways to show that the joining of midpoints created a parallelogram support his conjecture? Could he have used the same way each time? Would using one way strengthen the conjecture?
• What pattern did Tracey notice that led to her conjecture? How do the attributes of the shapes Tracey has focused upon differ from those that Marc noticed?
• Is there another pattern that might have been noticed from Marc’s work? From Tracey’s work?
• Would Tracey’s conjecture fit Marc’s work? Would Marc’s conjecture fit Tracey’s work?

Sample Answers to Your Turn Questions

Example 1: From the evidence given, a conjecture that August is the driest month of the year is reasonable. For the 5 years of data, August has the least rainfall: 121.7 mm.

Background
Weather Conjectures
Long before weather forecasts based on weather stations and satellites were developed, people had to rely on patterns identified from observation of the environment to make predictions about the weather. For example:

• Animal behaviour: First Nations peoples predicted spring by watching for migratory birds. If smaller birds are spotted, it is a sign that spring is right around the corner. When the crow is spotted, it is a sign that winter is nearly over. Seagulls tend to stop flying and take refuge at the coast when a storm is coming. Turtles often search for higher ground when they expect a large amount of rain. (Turtles are more likely to be seen on roads as much as 1 to 2 days before rain.)
• Plant behaviour: Pine cone scales remain closed if the humidity is high, but open in dry air. The leaves of oak and maple trees tend to curl in high humidity.
• Personal: Many people can feel humidity, especially in their hair (it curls up and gets frizzy). High humidity tends to precede heavy rain.
**Example 2:** Yes. Jay’s conjecture is convincing because all the different combinations with positive and negative odd integers were used as samples. These three samples showed a pattern in their products, which Jay then tested with different integers. Jay’s conjecture was supported by this last sample.

No. Jay looked at only three cases before he made his conjecture, then tested it with only one more example. This is not a lot of evidence to base a conjecture on.

**Example 3:** It is possible to have two different conjectures about the same situation because different samples were used to develop the conjecture. Francesca used different values for the sizes of consecutive squares. When she examined her evidence, the common feature from her examples was different from the common feature that Steffan found from the evidence he had developed.

**Example 4:** a) The quadrilaterals that Marc and Tracey used were different. The quadrilaterals that Marc used were more varied than those that Tracey used.

b) Based on the evidence used, both conjectures seem valid. The conjecture that Marc developed would hold true for all of Tracey’s quadrilaterals, since a rhombus is a special type of parallelogram. But Tracey’s conjecture would not hold true for all of Marc’s quadrilaterals, since not all parallelograms are rhombuses.

**Sample Solution to the Key Question**

9. Sum of an odd integer and an even integer:

<table>
<thead>
<tr>
<th></th>
<th>Odd</th>
<th>Even</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+1</td>
<td>+2</td>
<td>+3</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td></td>
<td>+1</td>
<td>+2</td>
<td>+1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>+53</td>
<td>-14</td>
<td>+39</td>
</tr>
</tbody>
</table>

Based on the evidence gathered and the pattern in the sums, the following conjecture can be made: The sum of an odd integer and an even integer will always be an odd integer.

<table>
<thead>
<tr>
<th></th>
<th>Odd</th>
<th>Even</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+5</td>
<td>-6</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>-5</td>
<td>+6</td>
<td>+1</td>
</tr>
<tr>
<td></td>
<td>+100</td>
<td>-99</td>
<td>+1</td>
</tr>
</tbody>
</table>

**Closing (Pairs, Class), page 15**

Question 19 gives students an opportunity to make connections among the terms conjecture, inference, and hypothesis. Arguments can be developed to support the two given opinions. Allow students to explore the nuances of meaning among these terms. Use reference resources and students’ knowledge of these terms to support students’ understanding of how these terms are similar and how both Lou’s and Sasha’s opinions are valid.
### Assessment and Differentiating Instruction

#### What You Will See Students Doing...

<table>
<thead>
<tr>
<th>When students understand...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students make conjectures that consider the patterns in the information given and evidence gathered.</td>
<td>Students are unable to develop conjectures, or they make conjectures without seeing a pattern in the evidence, or they do not recognize the patterns within the evidence.</td>
</tr>
<tr>
<td>Students justify their conjectures by drawing upon specific evidence from the examples and by developing new examples to support their conjectures.</td>
<td>Students make faulty connections between the conjectures and the evidence.</td>
</tr>
</tbody>
</table>

#### Key Question 9

| Students correctly interpret the math language of the problem. | Students are unable to interpret the math language of the problem. |
| Students make a conjecture about the sum of an odd integer and an even integer, based on evidence they have gathered. | Students are uncertain how to gather evidence about the sum of an odd integer and an even integer. Students make a conjecture that is not based on the evidence. |
| Students justify their conjecture based on the evidence gathered and the specific patterns recognized. | Students’ justifications minimally connect to the evidence or do not make any connections to specific examples. |

#### Differentiating Instruction | How You Can Respond

**EXTRA SUPPORT**

1. If students have difficulty interpreting the language of the problem, review Example 2, its language, and the steps that were used to develop a conjecture.

   2. If students have difficulty seeing a pattern in the specific examples they try, suggest that they use a table to show their results for the specific examples. The table may help students focus upon the patterns in the evidence.

**EXTRA CHALLENGE**

1. Students could create their own problem to investigate by gathering data, making conjectures, and then testing their conjectures with more specific cases.

   2. Students could work in pairs to develop sets of data and conjectures on separate cards. These cards could be used in a concentration game.
**Math in Action**

Students can be invited to reflect on the discussion of the chapter opener when dealing with this problem. The similarities between the situation in the chapter opener and the situation here may encourage students to consider what each person saw in light of his or her perspective and experience during the accident. Various conjectures may be developed, but each needs to be linked to the evidence gathered.

**Sample Solution**

Conjectures:
- Witness at stop sign: Yellow car did not completely stop; blue car was speeding.
- Driver of blue car: I was driving 60 km/h; the yellow car did not stop.
- Driver of yellow car: I came to a full stop.
- Investigator: No brake marks were evident due to snow cover.

Conjecture about the cause of the accident: Driver of blue car was not familiar with the area, its speed limits, or its traffic patterns.

Evidence that supports the conjecture: Passenger in blue car was looking at a map at the time of the accident.

Three questions to ask:
- Investigator: What evidence showed slippery road conditions?
- Witness: Which car was in the intersection first? In which direction were you crossing the street?

The cause of this accident cannot be proved, since there are conflicting pieces of evidence. Each driver contradicts the other, and there is minimal corroboration for either driver's allegation.
1.2 EXPLORING THE VALIDITY OF CONJECTURES

Lesson at a Glance

Prerequisite Skills/Concepts

- Gather evidence to support or refute a conjecture.
- Use inductive reasoning to make a conjecture.

WNCP

Specific Outcome

Logical Reasoning

1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators

1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.7 Determine if a given argument is valid, and justify the reasoning.

Introducing the Lesson

(10 min)

To introduce a discussion about the validity of conjectures, present the following situation: We know that optical illusions trick our eyes into believing something that may not be valid. How do these optical illusions make us think that things are not as they are? What methods can be used to check the validity of the conjectures?

Caution: A web search for optical illusions will result in many examples of optical illusions that are different from those best suited to this lesson. Care needs to be exercised when using online resources, since some optical illusions may not be appropriate for classroom use.

Teaching and Learning

(35 to 45 min)

Explore the Math (Individual, Pairs, Class), page 16

Introduce the exploration by asking students to identify and record their first reaction to each optical illusion. Then, after students have recorded their
first reactions to all the illusions, ask them to look at the illusions again and determine whether they still have the same reactions. Note that there is a blackline master with the illusions on page 57 of this Teacher’s Resource.

After students complete the exploration, invite pairs of students to share the methods they used to test the validity of their conjectures.

**Sample Solution to Explore the Math**

**First image:** Diagonal $AB$ is longer than diagonal $BC$. (The two diagonals could be measured with a ruler to confirm that the two diagonals are the same length.)

**Second image:** The centre circles are different sizes; the circle on the left is smaller than the circle on the right. (Measurement could be used to validate the conjecture. If calipers are available, then the diameter of the two circles could be compared directly to confirm that both circles are the same size.)

**Third image:** The rows and columns of white and black shapes are not straight. (A straightedge could be used to validate the conjecture. By placing the straightedge across the figure for each row and column, the straightness could be confirmed.)

**Fourth image:** There are two triangles: one white and one edged with red. (Visual examination of the figure from a different perspective can show that there are no triangles in the figure.)

**Reflecting, page 16**

The questions that are posed invite students to refine their understanding of conjectures. The process of making a conjecture and then amending it, based on new information, is characteristic of inductive reasoning. Presenting a situation in which students are expected to make amendments to their conjecture, after they have gathered evidence that refutes its validity, encourages the realization that when new information becomes available, a new or modified conjecture may be needed.

**Sample Answers to Reflecting**

A. Both measurement and visual inspection helped to verify or discredit the conjectures.

B. My conjectures changed as follows after collecting more evidence:

- First image: Both diagonals are the same length.
- Second image: The centre circles of the figures are the same size.
- Third image: The rows and columns of white and black shapes are placed in straight lines.
- Fourth image: There are no triangles in the figure.

C. For these images, the revised conjectures hold true for the accuracy of the tools I used. I cannot be absolutely sure that my new conjectures are valid until the precision of the tools is considered.

**Math Background**

- The link between making a conjecture and gathering evidence to determine the validity of the conjecture promotes the development of strong justification for the conjecture.
- When evidence counters a conjecture, the conjecture may be revised to reflect this new information. Then more evidence may be gathered to support the revised conjecture.
- The link to other sciences and the revision of theories over time may be used as an analogy for students to consider.
Consolidation
(10 to 15 min)

Further Your Understanding, page 17
Students use strategies from the exploration and Lesson 1.1 to test conjectures they make about the situations presented. Students should be allowed to work in pairs for this section, since discussing their ideas will help them identify strategies for checking their conjectures and justifications.

In each question, students are asked to make a conjecture and then validate it. The first question is another example of an optical illusion, in which the tabletops are exactly the same. The second question presents a numeric pattern, and the third question presents a geometric pattern.

Assessment and Differentiating Instruction

<table>
<thead>
<tr>
<th>What You Will See Students Doing...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td>When students understand...</td>
<td>If students misunderstand...</td>
</tr>
<tr>
<td>Students make conjectures, gather evidence, and revise their conjectures.</td>
<td>Students do not know what steps they should follow to find support for their conjectures. Students will be reluctant to make conjectures based on a single image.</td>
</tr>
<tr>
<td>Students understand that conjectures may be changed to reflect more evidence.</td>
<td>Students are unable to revise a conjecture to reflect more evidence or to make the conjecture more reasonable or clear.</td>
</tr>
</tbody>
</table>

Differentiating Instruction | How You Can Respond

EXTRA SUPPORT
1. Encourage students to record their first impressions of the optical illusions. These first impressions can form the basis of their conjectures. For example, “My first impression of the third illusion is that the image bulges.” To help students refine their impressions and develop testable conjectures, ask questions such as these: What do you mean by “bulge”? How else can you describe what you see?

2. Students may need to have visual reminders about the steps they should follow to develop and then validate a conjecture. A table that summarizes these steps will provide a reminder of these steps.

EXTRA CHALLENGE
1. Students can find other optical illusions to share with the class.

2. Students can create their own optical illusion using black lines and a red circle. By limiting the elements in the task, students will need to think flexibly about how to solve the problem.
1.3 USING REASONING TO FIND A COUNTEREXAMPLE TO A CONJECTURE

Lesson at a Glance

Prerequisite Skills/Concepts
- Make conjectures.
- Gather evidence to support or refute a conjecture.
- Identify powers of 2, consecutive perfect squares, prime numbers, and multiples.
- Determine square roots and squares.

WNCP
Specific Outcome
Logical Reasoning
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators
1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.4 Provide and explain a counterexample to disprove a given conjecture.

Math Background
- In this lesson, students examine conjectures and identify counterexamples from the development of more evidence.
- Students develop the concept that conjectures are valid until a single exception is found. Conjectures may then be revised to accommodate the exception. If a conjecture cannot be revised to accommodate the exception, then a new conjecture must be developed.

GOAL
Invalidate a conjecture by finding a contradiction.

Student Book Pages 18–25

Preparation and Planning

Pacing
- 10 min Introduction
- 35–45 min Teaching and Learning
- 10–15 min Consolidation

Materials
- calculator
- ruler
- compass

Recommended Practice
Questions 3, 6, 9, 10, 12, 15

Key Question
Question 14

New Vocabulary/Symbols
counterexample

Mathematical Processes
- Communication
- Connections
- Problem Solving
- Reasoning

Nelson Website
http://www.nelson.com/math

Introducing the Lesson

(10 min)

Explore (Pairs, Class), page 18
The Explore problem can be assigned for pairs of students to complete. The problem provides an opportunity for students to analyze a conjecture and then gather further evidence as they search for a counterexample. After all
the pairs find a counterexample, discuss what strategies they used. Introduce the idea of changing the conjecture to represent the new evidence. A connection may be made to the sciences, since this process of making a conjecture based on the evidence available, finding a counterexample, and then refining the conjecture is how scientific theories are improved.

**Sample Solution to Explore**

The number words to 100 contain all the vowels except *a*.

<table>
<thead>
<tr>
<th>zero</th>
<th>ten</th>
<th>twenty</th>
</tr>
</thead>
<tbody>
<tr>
<td>one</td>
<td>eleven</td>
<td>thirty</td>
</tr>
<tr>
<td>two</td>
<td>twelve</td>
<td>forty</td>
</tr>
<tr>
<td>three</td>
<td>thirteen</td>
<td>fifty</td>
</tr>
<tr>
<td>four</td>
<td>fourteen</td>
<td>sixty</td>
</tr>
<tr>
<td>five</td>
<td>fifteen</td>
<td>seventy</td>
</tr>
<tr>
<td>six</td>
<td>sixteen</td>
<td>eighty</td>
</tr>
<tr>
<td>seven</td>
<td>seventeen</td>
<td>ninety</td>
</tr>
<tr>
<td>eight</td>
<td>eighteen</td>
<td>hundred</td>
</tr>
<tr>
<td>nine</td>
<td>nineteen</td>
<td></td>
</tr>
</tbody>
</table>

These number words are used for all the numbers to 999. The word *thousand* is the first number word that contains the vowel *a*.

**2 Teaching and Learning**

*(35 to 45 min)*

**Learn About the Math (Class, Pairs), pages 18 and 19**

*Example 1* presents a series of circles and the related table of values. As the example is discussed, ask these questions:

- Is Kerry’s conjecture reasonable?
- What other conjectures could be made, based on the evidence?
- How would you check the validity of Kerry’s conjecture?
- What steps would you take to check the validity of Kerry’s conjecture?

The term *counterexample* is introduced in this example.

**Reflecting, page 19**

Students could talk in pairs about the Reflecting questions before discussing them as a class. After the class discusses the answers to these questions, invite students to

1. consider how Kerry’s conjecture might be changed to fit the new evidence, and
2. identify what steps might be needed before revising the conjecture.
Sample Answers to Reflecting

A. I think Zohal started her samples with five points on the circle to continue the pattern in Kerry’s evidence. If there are regular increments in the pattern, then possible counterexamples in the lesser values might be found. This would avoid the need to work with greater numbers of points and the challenge of counting the resulting regions.

B. One counterexample is enough to disprove a conjecture because the counterexample shows a case when the conjecture is not valid. Once a counterexample is found, the conjecture is no longer valid.

Consolidation

(10 to 15 min)

Apply the Math (Class, Pairs), pages 19 to 21

Using the Solved Examples

Example 2 makes connections to Lesson 1.1, when two different conjectures were developed in response to the same situation—the difference between consecutive perfect squares. After studying the example, discuss why all conjectures are not valid and how more evidence may strengthen a conjecture but does not prove it.

Ask the students the following questions to help guide their reflections on the development of the two conjectures from Lesson 1.1 and the further testing of these conjectures.

- How did Francesca choose to gather her evidence? How did this evidence lead her to notice the pattern she did? What patterns did Francesca notice that led to her conjecture?
- How did Steffan gather his evidence? How did Steffan’s pattern of evidence development differ from Francesca’s? What patterns did Steffan notice? After studying the example, ask students to reflect on Francesca’s conjecture and her method of gathering evidence about the difference of consecutive squares. Based on the evidence she gathered, was her conjecture reasonable? What could she have done differently to lead her to a valid conjecture?
- Francesca’s conjecture is reasonable based on the evidence that she gathered. However, when further evidence was gathered, the conjecture was found to be invalid. Steffan’s conjecture is reasonable based on the evidence that he gathered. When further evidence was gathered, the conjecture was supported. Why is further evidence that supports Steffan’s conjecture not considered to be proof that it is true in all cases?

Example 3 presents a conjecture about a numeric pattern. This example introduces the idea of revising a conjecture after a counterexample has been found, showing how the revised conjecture might include the new evidence.
After discussing the examples, ask students to reflect on the following questions:

• What did you notice about the search for counterexamples? (systematic gathering of more evidence)

• How did a counterexample affect the conjecture?

• Could the conjecture be revised to accommodate the new evidence?

Sample Answers to Your Turn Questions

Example 2:  

a) \(8^2 - 7^2 = 15\)

15 is not a prime number.

b) I can’t find a counterexample to Steffan’s conjecture because Luke’s visualization presents a strong argument for the conjecture being valid in all cases. Even though Luke’s visualization does not prove the conjecture for all cases, it strengthens my belief that the pattern will be repeated in all cases.

Example 3:  

If Kublu had not found a counterexample at the 10th step, she could have still stopped there. With the quantity of evidence found to support the conjecture, and a two-digit number further validating the conjecture, the conjecture could be considered strongly supported. If she had wanted to do one more example, then it might have been logical to try a three-digit number to see if the conjecture was valid in that case.

Sample Solution to the Key Question

14. Conjecture: All natural numbers can be written as the sum of consecutive numbers.

I noticed that the sums Tim chose were not consecutive, so I started to fill in the gaps in Tim’s evidence.

\[
\begin{array}{c|c}
1 & 2 \\
0 + 1 & 1 + 1
\end{array}
\]

2 is a natural number, but it cannot be written as the sum of consecutive numbers. I disagree with Tim’s conjecture because 2 is a counterexample.

Closing (Pairs, Class), page 25

For question 18, ensure that students review and have examples of inductive reasoning, evidence, and counterexamples. As students begin to consider the relationships among these concepts, encourage them to connect with examples from other disciplines to support their explanations. As a class, discuss the relationships among these concepts to strengthen the understanding that case-by-case evidence in support of a conjecture does not mean that the conjecture has been proved.
## Assessment and Differentiating Instruction

### What You Will See Students Doing...

<table>
<thead>
<tr>
<th>When students understand...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students identify patterns in the evidence to develop conjectures.</td>
<td>Students are unable to identify a pattern.</td>
</tr>
<tr>
<td>Students, when they find a counterexample, consider whether the conjecture can be revised to accommodate the new information.</td>
<td>Students do not realize when a counterexample has been found.</td>
</tr>
<tr>
<td>Students can explain how a counterexample invalidates a conjecture.</td>
<td>Students cannot revise a conjecture to accommodate new information.</td>
</tr>
<tr>
<td>Students consider specific evidence supporting a conjecture as proof that a conjecture is true. Students do not make connections between a counterexample and the validity of a conjecture.</td>
<td></td>
</tr>
</tbody>
</table>

### Key Question 14

| Students approach the task in a systematic way, gathering evidence that will either support or refute the conjecture. | Students approach the task without an organized plan for gathering the evidence. They may choose samples at random, leading to more support for the conjecture. |
| Students justify their opinion using the counterexample found. | Students do not link finding a counterexample to the invalidation of a conjecture. |

### Differentiating Instruction | How You Can Respond

**EXTRA SUPPORT**

1. If students do not approach the task in an organized way, encourage them to check the strategies used to find counterexamples in the examples. As Pierre did in Example 2, organizing the information and then filling in the gaps may be helpful.

**EXTRA CHALLENGE**

1. Ask students to explore what would be reasonable as a range of specific cases to gather in a systematic way before considering a conjecture to be valid without proof. This information could be presented to the class for their acceptance.

2. Some students may benefit from the use of technology when testing the validity of conjectures. Spreadsheets help with calculating and organizing data related to number patterns. Dynamic geometry software is useful when dealing with conjectures involving geometric properties.
History Connection

Reasoning in Science

Students may choose to explore this concept in depth for their course project. They may identify a scientific theory that has significantly changed over time as more evidence became available. Both conjectures and scientific theories are revised based on counterexamples. Technology has been instrumental for identifying counterexamples and could also be the focus of a research project.

Answers to Prompts

A. The conjecture that Earth is the centre of the universe was refuted as scientists gathered evidence about the apparent motion of the Sun and the motions of the planets and their moons. The new evidence supported the heliocentric conjecture.

B. Inductive reasoning plays into our beliefs and understandings about our universe because the patterns we see in the natural world lead us to make conjectures about why these patterns exist. Since we are likely to notice these patterns on our own, we develop personal conjectures about the world and, until a counterexample is found, we continue to believe our conjectures. For example, in physiology, people have probably always known that a beating heart is necessary for life. Why it is necessary was subject to conjecture. The research of William Harvey and his predecessors and colleagues provided the observation that the heart pumps blood, leading to the modified conjecture that the circulation of blood is necessary for life.

Applying Problem-Solving Strategies

WNCP

Specific Outcomes

Logical Reasoning

1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]
2. Analyze puzzles and games that involve spatial reasoning, using problem solving strategies. [CN, PS, R, V]

Achievement Indicators

1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
2.1 Determine, explain and verify a strategy to solve a puzzle or to win a game.
2.2 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.
2.3 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Analyzing a Number Puzzle

A blackline master for the puzzle is provided on page 58 of this Teacher’s Resource. The puzzle requires students to use numerical reasoning and the patterns of evidence to develop a strategy for the solution. If students need help, suggest strategies such as guess and check, and looking for a pattern in the numbers of already-solved examples. As well, students could explore different polygons and their patterns.
Answers to Prompts

A.

a)  
\[ \begin{array}{c}
17 \\
9 \\
\end{array} \quad \begin{array}{c}
18 \\
19 \\
\end{array} \quad \begin{array}{c}
10 \\
\end{array} \]

b)  
\[ \begin{array}{c}
9 \\
6 \\
\end{array} \quad \begin{array}{c}
12 \\
15 \\
\end{array} \quad \begin{array}{c}
9 \\
\end{array} \]

c)  
\[ \begin{array}{c}
38 \\
29 \\
\end{array} \quad \begin{array}{c}
48 \\
68 \\
\end{array} \quad \begin{array}{c}
39 \\
\end{array} \]

C. I noticed these patterns:
- When the square numbers are consecutive, so are the circle numbers.
- When the square numbers are evenly sequenced, so are the circle numbers.
- When the square numbers are all even, the circle numbers are either all odd or all even.
- The sum of the square number and the circle number opposite are the same for that arithmagon.
- The sum of the square numbers divided by 2 is equal to the sum of a square number and its opposite circle number.
- The sum of the square numbers divided by 2 is equal to the sum of the circle numbers.

D. The relationship between the circle numbers and the opposite square numbers is that their sums are the same for each arithmagon. Another relationship is that the greatest square value is opposite the least circle value, the least square value is opposite the greatest circle value, and the median square value is opposite the median circle value.

E. Answers may vary, e.g., guess and check was the strategy I used.

F. Answers may vary, e.g., arithmagon a) was the easiest because the square numbers were consecutive numbers, so the circle numbers were also consecutive numbers. From the example, the circle number opposite the median square number was half the square number. I used this pattern to say that 18 was the median square number, so the circle number opposite was 9. Once I had 9, I could work my way around to determine the other circle numbers.
1.4 PROVING CONJECTURES: DEDUCTIVE REASONING

Lesson at a Glance

Prerequisite Skills/Concepts

- Make conjectures.
- Gather evidence to support or refute a conjecture.
- Revise a conjecture if a counterexample is found.
- Represent a situation algebraically.
- Simplify, expand, and evaluate algebraic expressions.
- Identify consecutive perfect squares and multiples.
- Interpret Venn diagrams.
- Identify vertically opposite angles and supplementary angles in intersecting lines.

WNCP

Specific Outcome

Logical Reasoning

1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators

1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.3 Compare, using examples, inductive and deductive reasoning.
1.4 Provide and explain a counterexample to disprove a given conjecture.
1.5 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies or algebraic number tricks
1.6 Prove a conjecture, using deductive reasoning (not limited to two column proofs).
1.7 Determine if a given argument is valid, and justify the reasoning.
1.9 Solve a contextual problem that involves inductive or deductive reasoning.

GOAL

Prove mathematical statements using a logical argument.

Student Book Pages 27–33

Pacing

<table>
<thead>
<tr>
<th>Preparation and Planning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pacing</td>
</tr>
<tr>
<td>10 min Introduction</td>
</tr>
<tr>
<td>35–45 min Teaching and Learning</td>
</tr>
<tr>
<td>10–15 min Consolidation</td>
</tr>
</tbody>
</table>

Materials

- calculator
- ruler

Recommended Practice

Questions: 4, 7, 8, 15

Key Question

Question 10

New Vocabulary/Symbols

- proof
- generalization
- deductive reasoning
- transitive property
- two-column proof

Mathematical Processes

- Communication
- Connections
- Problem Solving
- Reasoning

Nelson Website

http://www.nelson.com/math

Math Background

- This lesson presents the first two-column proof. The formal structure of the proof and the language used should be considered explicitly as a class.
- The difference between a two-column proof and a logical argument that presents proof of a conjecture should be explained.
Introducing the Lesson

(10 min)

Explore (Pairs, Class), page 27

The Explore problem leads students to consider the differences among a conjecture, evidence, and proof. Students may gather a vast quantity of evidence to support a conjecture, but this evidence only strengthens the validity of the conjecture. To prove a conjecture, all cases must be considered. The connections among a conjecture, an inference, and a scientific hypothesis could be revisited to explore their relationship to the mathematical proof of a conjecture.

Sample Solution to Explore

The conjecture “All teens like music” can be supported inductively by collecting more evidence. A questionnaire or an online survey could be tools to help gather the evidence. The conjecture cannot be proved because it is impossible to ask all teens. However, the conjecture can be refuted with one counterexample: a student who dislikes music.

Teaching and Learning

(35 to 45 min)

Learn About the Math (Class), page 27

Example 1 links a conjecture with some supporting evidence to the mathematical argument for proof of all cases. As the example is discussed, ask questions such as these:

• How could Pat have used different expressions to represent the five consecutive integers in her proof?
• How would expressing the five consecutive integers in a different way change the proof?

Reflecting, page 28

The term deductive reasoning could be introduced by comparing and contrasting inductive and deductive reasoning. Exploring their differences through examples and reflection on previous lessons will strengthen students’ understanding of the attributes of each. It will also strengthen students’ understanding of the concept that one form of reasoning is not subordinate to the other—they work together.

Sample Answers to Reflecting

A. Jon used inductive reasoning to make his conjecture. He analyzed a pattern he noticed and developed a conjecture about this pattern.

Math Background

- A formal proof has a specific structure to present explicit links between statements and their justification. The justification uses relationships known to be valid (previously proved or accepted as axioms).
- Prior to developing their own proofs, all students may benefit from an exploration of relationships they already know to be valid, such as the Pythagorean theorem and the sum of the measures of complementary angles.
- In mathematics, once a conjecture has been proven it becomes a theorem. Theorems can then be used in proofs of other conjectures.
B. Pat’s reasoning differed from Jon’s because she represented any five consecutive integers with variables, not with specific sets of five consecutive integers as Jon did. Because Pat’s deductive reasoning showed that the conjecture was true for any five consecutive integers, she proved that the conjecture was true for all cases. Jon was only able to say that the conjecture was true for the specific sets of consecutive integers that he sampled.

Consolidation
(10 to 15 min)

Apply the Math (Class, Pairs), pages 28 to 30

Using the Solved Examples

*Example 2* revisits the conjecture that the difference between consecutive perfect squares is an odd number. Reminding students about the last step in Luke’s support for Steffan’s conjecture (visualizing) may strengthen Gord’s algebraic proof. Allowing students the chance to consider the *Your Turn* problem individually before discussing it in pairs may encourage them to form their own opinions.

*Example 3* employs deductive reasoning to determine a logical conclusion. This type of example involves relationships of sets within sets and shows how a conclusion may be made by examining these relationships.

*Example 4* is the first example with a two-column proof (further developed in Chapter 2). To scaffold the learning experience for the next chapter, discuss how a formal two-column proof is formatted and what types of statements and explanations are used. Have students work in pairs to complete the *Your Turn* task, to support the development of understanding about the structure of a two-column proof.

*Example 5* uses deductive reasoning to prove the divisibility rule for 3. This example may need detailed examination to allow full understanding. The *Your Turn* task should be assigned as a paired task. The discussion between partners as they develop their proof should help them support their reasoning.

Sample Answers to Your Turn Questions

**Example 2:** Luke’s visualization may have helped Gord understand that the difference is always going to have two equal sets of tiles, plus one more. Since two equal sets will always represent an even number \((2n)\) is an even number), the additional single tile will always make the difference odd.

**Example 3:** I can deduce that Inez is building muscle. The other connections from the given statements lead from weight-lifting, but I cannot deduce that Inez is either strong or has improved balance. The act of building muscle does not mean...
that you have currently gained the muscle needed for strength and improved balance.

**Example 4:**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Justification</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>∠AEC + ∠AED = 180°</td>
<td>Supplementary angles</td>
<td>The measures of two angles that lie on the same straight line have a sum of 180°.</td>
</tr>
<tr>
<td>∠AED = 180° − ∠AEC</td>
<td>Subtraction property</td>
<td></td>
</tr>
<tr>
<td>∠CEB + ∠AEC = 180°</td>
<td>Supplementary angles</td>
<td></td>
</tr>
<tr>
<td>∠CEB = 180° − ∠AEC</td>
<td>Subtraction property</td>
<td></td>
</tr>
<tr>
<td>∠AED = ∠CEB</td>
<td>Transitive property</td>
<td>Two quantities that are equal to the same quantity are equal to each other. In this example, both angle measures are equal to 180° − ∠AEC.</td>
</tr>
</tbody>
</table>

**Example 5:**

\[
\begin{align*}
abc &= 100a + 10b + c \\
abc &= (99a + a) + (9b + b) + c
\end{align*}
\]

I let \(abc\) represent any three-digit number. Then I wrote \(abc\) in expanded form, decomposing 100a and 10b into equivalent sums.

\[
\begin{align*}
abc &= (99a + 9b) + (a + b + c) \\
abc &= 3(33a + 3b) + (a + b + c)
\end{align*}
\]

I grouped the terms that had 9 as a factor.

\[
abc \text{ will be divisible by 3 only when } (a + b + c) \text{ is divisible by 3.}
\]

\[
3(33a + 3b) \text{ is always divisible by 3 because 3 is a factor.}
\]

**Sample Solution to the Key Question**

**10.** Let \((2n + 1)\) represent any odd number.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2n + 1)^2 = (2n + 1)(2n + 1))</td>
<td>I expanded the expression.</td>
</tr>
<tr>
<td>((2n + 1)^2 = 4n^2 + 2n + 2n + 1)</td>
<td>I expanded the expression.</td>
</tr>
<tr>
<td>((2n + 1)^2 = 4n^2 + 4n + 1)</td>
<td>I combined like terms.</td>
</tr>
<tr>
<td>((2n + 1)^2 = 2(2n^2 + 2n) + 1)</td>
<td>I grouped the terms that had 2 as a factor.</td>
</tr>
<tr>
<td></td>
<td>Since two times any number is an even number, the square of any odd number will always be an even number plus 1, which is an odd number.</td>
</tr>
</tbody>
</table>
Closing (Pairs, Small Groups, Class), page 33

For question 17, have students work in pairs or small groups to develop their argument. Invite students to include the terms *inductive reasoning*, *evidence*, *deductive reasoning*, *generalization*, and *mathematical proof*. After the pairs or groups have completed the question, discuss their ideas as a class. Students should note that each of the three examples has weaknesses that could be strengthened.

### Assessment and Differentiating Instruction

<table>
<thead>
<tr>
<th>What You Will See Students Doing...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>When students understand...</strong></td>
<td></td>
</tr>
<tr>
<td>Students use the new terms correctly.</td>
<td>Students either avoid using the new terms or use the terms incorrectly.</td>
</tr>
<tr>
<td>Students apply their knowledge of inductive and deductive reasoning appropriately.</td>
<td>Students are unable to differentiate between examples of inductive and deductive reasoning.</td>
</tr>
<tr>
<td>Students prove a conjecture.</td>
<td>Students are unable to use deductive reasoning to prove a conjecture.</td>
</tr>
</tbody>
</table>

**Key Question 10**

Students develop an algebraic expression to reflect the problem and then simplify their expression to prove the conjecture.

Students explain the simplification clearly and accurately.

**Students are unable to develop an algebraic expression.**

**Students are unable to explain the steps in the simplification clearly and accurately.**

### Differentiating Instruction | How You Can Respond

**EXTRA SUPPORT**

1. If students have difficulty translating the problem into an algebraic expression, scaffolding with questions may help them build the expression. Ask questions such as these: How can any integer be represented? How can an even integer be represented? How can this representation be changed to show an odd integer?

**EXTRA CHALLENGE**

1. Ask students to review conjectures they considered to be valid in previous lessons and develop proofs for these conjectures. This task could be done in pairs so that conversation becomes part of the process, for both choosing the conjectures and developing the proof.
MID-CHAPTER REVIEW

Using the Frequently Asked Questions

Have students keep their Student Books closed while you display the Frequently Asked Questions without the answers. Discuss the questions as a class, and use the discussion to draw out what students think is a good answer to each question. Then have students compare the class answers with the answers on Student Book page 34. Invite students to consider how both of the two types of reasoning are important in mathematics and other disciplines. Encourage students to revise any conjectures that were disproved in Lesson 1.3.

Using the Mid-Chapter Review

Ask students to reflect individually on the goals of the lessons completed so far. Ask students to identify, on their own, the lesson or goal that was most challenging or any lessons that need more explanation to improve their understanding. Then have students work in pairs to develop questions that, when answered, would improve their understanding.

Review the topics from the first part of the chapter. Respond to the questions that students have developed. Use the Practising questions to reinforce students’ knowledge, understanding, and skills, so that students are prepared for the second half of the chapter. Assign the Practising questions for in-class work and for homework.

Student Book Pages 34–35

<table>
<thead>
<tr>
<th>Question</th>
<th>Curriculum</th>
<th>Processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LR1.1</td>
<td>CN, R</td>
</tr>
<tr>
<td>2</td>
<td>LR1.1</td>
<td>CN, PS, R</td>
</tr>
<tr>
<td>3</td>
<td>LR1.1</td>
<td>CN, PS, R</td>
</tr>
<tr>
<td>4</td>
<td>LR1.1, LR1.7</td>
<td>PS, R</td>
</tr>
<tr>
<td>5</td>
<td>LR1.4</td>
<td>CN, R</td>
</tr>
<tr>
<td>6</td>
<td>LR1.4, LR1.7</td>
<td>CN, PS, R</td>
</tr>
<tr>
<td>7</td>
<td>LR1.2, LR1.3, LR1.7</td>
<td>C, CN, R</td>
</tr>
<tr>
<td>8</td>
<td>LR1.1, LR1.5, LR1.6</td>
<td>PS, R</td>
</tr>
<tr>
<td>9</td>
<td>LR1.5, LR1.6</td>
<td>PS, R</td>
</tr>
<tr>
<td>10</td>
<td>LR1.6, LR1.7</td>
<td>CN, PS, R</td>
</tr>
<tr>
<td>11</td>
<td>LR1.6</td>
<td>PS, R</td>
</tr>
</tbody>
</table>
1.5 PROOFS THAT ARE NOT VALID

Lesson at a Glance

Prerequisite Skills/Concepts
• Present a logical argument using inductive and deductive reasoning.
• Apply and interpret algebraic reasoning and proofs.
• Simplify, expand, and evaluate algebraic expressions.
• Solve algebraic equations.
• Factor algebraic expressions, including a difference of squares.

WNCP
Specific Outcome
Logical Reasoning
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.4 Provide and explain a counterexample to disprove a given conjecture.
1.5 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies or algebraic number tricks.
1.7 Determine if a given argument is valid, and justify the reasoning.
1.8 Identify errors in a given proof; e.g., a proof that ends with $2 = 1$.

GOAL
Identify errors in proofs.

Introducing the Lesson
(10 min)

Explore (Groups, Class), page 36
Consider the following statement: “There are three errors in this statement.” Is the statement true?

Students explore the concept of errors in proofs by examining the statement and deciding whether it is true. This is an example of circular reasoning. It shows how invalid proofs may seem correct, but the initial statement is in doubt. Discuss with students how a statement may be circular.

Sample Solution to Explore
There are only two spelling errors in the statement, not three, so the statement is invalid. If the statement is invalid, however, the statement itself is an error, making a total of three errors in the statement. Because the statement contains three errors, it is valid. But a statement cannot be both valid and invalid.
Teaching and Learning
(35 to 45 min)

Investigate the Math (Pairs, Class), page 36
This investigation allows students to examine the validity of a proof and develop strategies for examining a proof that common sense tells them cannot be true. Invite students to predict where they think the error in the proof occurs. Have pairs of students use different scales for one square tile, to emphasize the need for precision and to allow students to observe where the error occurs. After the pairs work through the prompts, encourage them to share their answers with other pairs in the class.

Sample Answers to Prompts

C., D.

E. No. There is a gap along the diagonal of the rectangle, which shows that the area of the rectangle is not 65. Perhaps the thick black outline of the shapes fills in this gap in the diagram on page 36. But when I recreated the diagram, I could see a gap.

Reflecting, page 36
The Reflecting questions can be discussed in groups of three and then as a class. Have all the groups report their answers, with a different student from each group reporting each answer.

Sample Answers to Reflecting

F. Any overlap or empty space suggests that there is error in the proof. If the pieces had overlapped in any way, this would have indicated that the area of the rectangle was less than the area of the square. The empty space indicates that the area of the rectangle is actually greater than the area of the square.

G. The colours of the figures and their black outlines are like an optical illusion. My eyes tell me that both figures are made with the same pieces, but I know that 64 ≠ 65. When I look at the figures, the pieces seem to be identical.

H. Errors in construction come from a lack of care and precision. By enlarging the size of the unit square, errors may be easier to avoid and easier to recognize.
Consolidation
(10 to 15 min)

Apply the Math (Class), pages 37 to 41

Using the Solved Examples

In Example 1, students are presented with another example of circular reasoning. In this example, the error is obvious in the first statement. Discuss how the argument could be made valid.

Example 2 provides the first algebraic example of a false proof. Helping students identify common errors, such as dividing by zero, will allow them to see that anything can be “proved” in a proof with a false statement.

Example 3 provides a different example of circular reasoning, this time using algebra. The argument is based on a false first statement, making the whole argument invalid.

Example 4 uses a number trick to have students examine an algebraic proof for errors. By finding the error in the proof, students may develop more awareness of where to look in their own proofs for errors.

Example 5 presents an argument about money. The assumption that money and decimals are the same provides the core of this example’s falseness.

Sample Answers to Your Turn Questions

Example 1: The error is in the second statement. Not all high school students dislike cooking.

Example 2: Suppose that \(a + b = c\).

\[
\begin{array}{|c|}
\hline
\text{The statement can be written as} & 65a - 64a + 65b - 64b = 65c - 64c \\
\hline
\text{After reorganizing, it becomes} & 65a + 65b - 65c = 64a + 64b - 64c \\
\hline
\text{Using the distributive property,} & 65(a + b - c) = 64(a + b - c) \\
\hline
\text{Dividing both sides by} \ (a + b - c), & 65 = 64 \\
\hline
\end{array}
\]

Example 3: An error in a premise is like a counterexample because a single error invalidates the argument, just as a single counterexample makes a conjecture invalid.

Example 4: Hossai’s number trick will work for every number because the proof uses \(n\) as any number and results in the number 5.

Example 5: Yes. Grant explains that squares of a currency unit do not make sense, which is what Jean is suggesting in her proof.

Background

Hayley Wickenheiser is a well-known hockey player. She was the first female, full-time, professional hockey player in a position other than goalie. She has represented Canada at the Olympics and at the Women’s World Hockey Championships, bringing home numerous medals.
Sample Solution to the Key Question

7. Let \( a = b \). This premise could be true.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>So, ( a^2 = ab )</td>
<td>If the premise is valid, then this equation is also valid.</td>
</tr>
<tr>
<td>( a^2 + a^2 = a^2 + ab )</td>
<td>Adding the same quantity to equal values keeps the equation valid.</td>
</tr>
<tr>
<td>( 2a^2 = a^2 + ab )</td>
<td>Combining like terms is valid.</td>
</tr>
<tr>
<td>( 2a^2 - 2ab = a^2 + ab - 2ab )</td>
<td>Subtracting equal values from both sides of the equation keeps the equation valid.</td>
</tr>
<tr>
<td>( 2a^2 - 2ab = a^2 - ab )</td>
<td>Combining like terms is valid.</td>
</tr>
<tr>
<td>Rewriting this as ( 2(a^2 - ab) = 1(a^2 - ab) ).</td>
<td>Factoring does not change the equality.</td>
</tr>
<tr>
<td>Dividing both sides by ( a^2 - ab ), we get ( 2 = 1 ).</td>
<td>This step is incorrect. If ( a^2 = ab ), then ( a^2 - ab = 0 ). Division by zero is undefined.</td>
</tr>
</tbody>
</table>

Closing (Pairs, Class), page 44

Encourage students to be specific in their discussion of question 8, using examples they have encountered in this lesson and in their own experiences. Students’ discussion could be summarized in a table. Invite students to express their opinions, but make sure that their opinions are justified.

For example, I looked at Practising Question 3 where the false proof states that \( 2 = 0 \). This is completely unreasonable. However, when the algebraic proof is followed, it appears that each step is reasonable. One of the steps has to be illogical. In this case, it is the division by zero that is masked by \( (a - b) \).

**Summary:** There seem to be typical kinds of errors—for example, division by zero or errors in the application of order of operations in algebra, invalid assumptions in logical arguments, and inaccuracy in drawing in geometry. Once an error is introduced, any conclusion derived from that basis is not valid.
## Assessment and Differentiating Instruction

<table>
<thead>
<tr>
<th>What You Will See Students Doing...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>When students understand...</strong></td>
<td><strong>If students misunderstand...</strong></td>
</tr>
<tr>
<td>Students are systematic and analytical in their examination of the proofs.</td>
<td>Students have difficulty being systematic in their analysis of the reasoning and proof.</td>
</tr>
<tr>
<td>Students identify errors, explain them clearly, and correct them.</td>
<td>Students are unable to explain where an error is and why it is an error.</td>
</tr>
<tr>
<td>Students identify types of errors that are common in false proofs.</td>
<td>Students do not categorize errors.</td>
</tr>
</tbody>
</table>

**Key Question 7**

Because the proof is algebraic, students initially look for two common types of errors: division by zero and order of operations.

Students systematically review the whole proof, recording the statements that are valid until the error is found.

Students are unable to identify the error, even when prompted to look for common errors.

Students do not review the whole proof systematically or analytically, which may cause them to miss the error.

### Differentiating Instruction | How You Can Respond

**EXTRA SUPPORT**

1. If students have difficulty identifying the errors in the proofs, review the examples in Apply the Math. Draw students’ attention to the example that models the type of error they are having difficulty identifying. Encourage students to remember the steps where errors commonly occur, and invite them to make connections between patterns in the proofs. For example, when an algebraic proof has a step with division, remind them to check if the divisor is equal to zero.

**EXTRA CHALLENGE**

1. Students may recognize patterns within algebraic false proofs. Ask students to create a new false proof that is modelled after one of the false proofs in this chapter. Ideas may be drawn from the examples or the Practising questions. Have students exchange false proofs with a partner, so the partner can try to identify the error.

2. If students have difficulty identifying the error in a proof, ask them to identify statements or parts of statements that they can confidently say are either valid or suspect. For example, statements that include words such as *every, all, or none* invite counterexamples.
1.6 REASONING TO SOLVE PROBLEMS

Lesson at a Glance

**Prerequisite Skills/Concepts**

- Make conjectures.
- Gather evidence to support or refute a conjecture.
- Revise a conjecture if a counterexample is found.
- Present a logical argument using inductive and deductive reasoning.
- Apply and interpret algebraic reasoning and proofs.
- Simplify, expand, and evaluate algebraic expressions.
- Solve algebraic equations.

**WNCP**

**Specific Outcome**

**Logical Reasoning**

1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

**Achievement Indicators**

1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.3 Compare, using examples, inductive and deductive reasoning.
1.4 Provide and explain a counterexample to disprove a given conjecture.
1.7 Determine if a given argument is valid, and justify the reasoning.
1.9 Solve a contextual problem that involves inductive or deductive reasoning.

**Introducing the Lesson**

(10 min)

**Explore (Pairs, Class), page 45**

Have students discuss the Explore problem in pairs initially. Invite students to consider what would be more important: light or heat. Suggest that students consider different contexts for the cabin. Is the cabin in the north or south? Is the season summer or winter?

**Sample Solution to Explore**

I would light the match first. If I didn’t, I couldn’t light any of the other items. I would light the candle next, since it would stay lit for longer than the match and would allow me to light the other two items. Also, it’s less likely that I would make an error or fail when lighting the candle. The lantern and the stove would be more difficult to light.
In the opening problem, clear examples of inductive and deductive reasoning are used. These examples reiterate the differences between the two types of reasoning. Asking students to work in small groups, with a distinct role for each group member (such as recorder, reporter, lead, or timer), allows all students to be involved. With the class, discuss how an organized approach to both inductive and deductive reasoning helps to structure explanations.

**Sample Answers to Prompts**

**A.** Emma might have chosen the four values because each value represents a different attribute. One value is positive, another is negative, another is zero, and the last is a larger number. With this variety, Emma might have thought that she had sampled sufficiently from the range of possible values.

**B.** The explanation does not include reasons for each step, nor does it show what each step looks like. It provides only a summary.

**C.** Conjecture: The resulting value will always be three times the starting value. Justification and explanation:

<table>
<thead>
<tr>
<th>Let $d$ represent any number.</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply by 6.</td>
<td>$6d$</td>
</tr>
<tr>
<td>Add 4.</td>
<td>$6d + 4$</td>
</tr>
<tr>
<td>Divide by 2.</td>
<td>$\frac{(6d + 4)}{2} = 3d + 2$</td>
</tr>
<tr>
<td>Subtract 2.</td>
<td>$3d + 2 - 2 = 3d$</td>
</tr>
<tr>
<td>The resulting value is three times the starting value.</td>
<td>$3d$</td>
</tr>
</tbody>
</table>

**Reflecting, page 46**

Students should work in pairs to explore their explanations and opinions, before sharing with the whole class.

**Sample Answers to Reflecting**

**D.** Understanding the mathematics represented by both symbols and words makes it easy to explain. For example, because I know that doubling a number means multiplying by 2, I can represent the words as $2x$.

**E.** A clear explanation ensures that the person who is reading it will follow your reasoning all the way through. If you miss steps, then the reader won’t understand or may reject your argument as invalid. If you don’t use precise language, diagrams, or algebra, then the reader may not understand.
Consolidation
(10 to 15 min)

Apply the Math (Class, Pairs), pages 46 to 48

Using the Solved Examples

In Example 1, students are encouraged to examine the type of reasoning used to solve the problem. Then, in the Your Turn question, students reflect on the characteristics of each type of reasoning and apply this knowledge to a new situation.

To direct students’ thinking about Example 1, introduce an activity based on traditional sharing circles. In small groups of four or five, students can share a piece of information about themselves, once with each person in the group. When the activity is done, ask students to count how many times pairs shared information and if there is a pattern to how information was shared.

In Example 2, students are presented with a logic problem and asked to solve it based on given information.

Sample Answers to Your Turn Questions

Example 1: Kim used inductive reasoning. To solve the problem, Kim determined the new number of handshakes based on the pattern identified in the first two cases. I know that Kim used inductive reasoning because the result was specific to this number of people, not a generalization that would be true for any number of people.

Example 2: Vicky used deductive reasoning. She used the given information to deduce the seating arrangements. The language in her explanation followed the pattern of if . . . then statements, which may be present in deductive reasoning.

Sample Solution to the Key Question

10. a) The envelope marked 8 could not possibly have the pair of cards (6, 2). For the greatest sum of 14, the only possible values are (8, 6) and (9, 5). If the envelope labelled 14 uses the 6, then no other envelope can have it. If the envelope labelled 14 does not have the 6, then it must have the combination (9, 5). The only possible values for the envelope that is labelled 13 are (9, 4), (8, 5), and (7, 6). But the envelope labelled 14 has the 9 and 5 cards in it, making the combinations (9, 4) and (8, 5) impossible. The only other possibility for the envelope labelled 13 is (7, 6), making it impossible for any other envelope to have the 6 in it.

b) The reasoning I used was deductive, because it involved a series of related steps: if one statement is valid, then the next statement must also be valid.
Practising, pages 49 to 51

12. Answers may vary. Students may consider various factors: e.g., placement of the coaches, whether it’s important that some students have their backs to others (for example, if they’re sitting inside a closed shape or in rows). In order to arrive at a “best” solution, students will have to agree on all of the factors they have raised in developing their individual solutions.

16. Students may become stuck by thinking about moving a pail, instead of about moving the water inside a pail. (The solution is to pour the contents of the second pail into the fifth pail.) Suggest that students who are stuck consider the pattern of full and empty, rather than moving a pail.

Closing (Pairs, Class), page 51

Question 17 gives students an opportunity to reiterate the characteristics of inductive and deductive reasoning. The patterns of these characteristics may be summarized by examining questions from the lesson.

Assessment and Differentiating Instruction

<table>
<thead>
<tr>
<th>What You Will See Students Doing...</th>
<th>If students misunderstand...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>When students understand...</strong></td>
<td></td>
</tr>
<tr>
<td>Students are able to differentiate between the types of reasoning.</td>
<td>Students are unable to identify the differences between inductive and deductive reasoning.</td>
</tr>
<tr>
<td>Students can apply reasoning to solve the problem.</td>
<td>Students have difficulty applying reasoning to solve problems.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Key Question 10</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Students organize an approach for solving the problem that includes both inductive and deductive reasoning.</td>
<td>Students are less likely to use an organized approach to problem solving. The strategy of guess and check may be used without reasoning or organization.</td>
</tr>
<tr>
<td>Students determine the solution and explain the reasons for any conclusions.</td>
<td>Students may determine the solution but not be able to apply deductive reasoning to explain the solution.</td>
</tr>
</tbody>
</table>

**Differentiating Instruction | How You Can Respond**

**EXTRA SUPPORT**
1. If students have difficulty starting a problem, suggest that they read the problem again to identify possible starting points and paths to a solution. For example, in the key question, students could consider possible cards for the other four envelopes.

**EXTRA CHALLENGE**
1. Encourage students to create their own problem for another student to solve.

2. Encourage students to model the problem. For example, in the key question, making cards to manipulate may help them with the solution.
1.7 ANALYZING PUZZLES AND GAMES

Lesson at a Glance

GOAL
Determine, explain, and verify a reasoning strategy to solve a puzzle or win a game.

Prerequisite Skills/Concepts
• Make conjectures.
• Gather evidence to support or refute a conjecture.
• Revise a conjecture if a counterexample is found.
• Present a logical argument using inductive and deductive reasoning.
• Apply and interpret algebraic reasoning and proofs.

WNCP
Specific Outcomes
Logical Reasoning
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]
2. Analyze puzzles and games that involve spatial reasoning, using problem solving strategies. [CN, PS, R, V]

Achievement Indicators
1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.2 Explain why inductive reasoning may lead to a false conjecture.
1.3 Compare, using examples, inductive and deductive reasoning.
1.4 Provide and explain a counterexample to disprove a given conjecture.
1.7 Determine if a given argument is valid, and justify the reasoning.
1.9 Solve a contextual problem that involves inductive or deductive reasoning.

2.1 Determine, explain and verify a strategy to solve a puzzle or to win a game.
2.2 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.
2.3 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Background
- Tic-tac-toe is known by many names, such as noughts and crosses or three-in-a-row. This game has been played for centuries, possibly as far back as 1300 BCE in Egypt. Grid games like tic-tac-toe have been recorded in carvings from ancient Roman times.

Math Background
- Winning a game involves looking for patterns and then developing a strategy to fit these patterns. Sometimes these patterns are numerical, such as those in magic squares and Sudoku. Sometimes these patterns are spatial, such as those in tic-tac-toe and checkers.
- Changing a rule or condition for a game may require a shift in the strategy. For example, if the winner of the game in Example 2 is the player who has no toothpicks left to pick up, the strategy for winning changes.
Introducing the Lesson

10 min

Explore (Groups of Three, Class), page 52
The Explore problem focuses on analyzing a game. Students may play the game to help them decide whether it is fair and which student they would prefer to be. In their analysis, students are encouraged to consider the possible outcomes. If students complete this problem quickly, they could consider how the game would change if three coins were used.

Sample Solution to Explore
Answers may vary, e.g., I would prefer to be student 3 since there are two chances of getting a head and a tail for every coin toss, but there is only one chance of getting both heads or both tails. The probability of student 3 winning is 2 : 4, but the probability of either student 1 or student 2 winning is 1 : 4.

Teaching and Learning

35 to 45 min

Investigate the Math (Groups of Three, Class), page 52
Students analyze a leapfrog puzzle, with the aim of determining the minimum number of moves needed to complete the puzzle. Through inductive reasoning, students explore the minimum possible number of moves.

Sample Answers to Prompts
A. In a group of three, two students can manage the movement of the counters while the third student can count the moves.
C. It took 35 moves to make the switch.

Reflecting (Small Groups, Class), page 52
The Reflecting questions can be discussed first in small groups and then as a class. Allowing students an opportunity to clarify their thinking in a small group encourages them to try out different possibilities and practise their reasoning. In the class discussion, encourage students to reflect on the knowledge, understanding, and skills they have developed throughout this chapter.

Sample Answers to Reflecting
D. As a group, we tried the puzzle three times. We found that 35 moves was the fewest number of moves we needed to complete the switch.
E. To solve this puzzle, we used inductive reasoning. We tried the puzzle using 1, 2, 3, 4, then 5 counters on each side to see if there was a pattern. The table shows that there is.

<table>
<thead>
<tr>
<th>Number of Counters of each Colour, n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Number of Moves</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>24</td>
<td>35</td>
</tr>
</tbody>
</table>
F. 3 = (1)(3), 8 = (2)(4), 15 = (3)(5), 24 = (4)(6), and 35 = (5)(7). In each case the minimum number of moves can be written as the product, \( n(n + 2) \). I predict that we would need a minimum of 48 moves with six counters: 6(8) = 48.

G. I used inductive reasoning because I analyzed the result of several cases and made a prediction based on this result. My prediction does not prove the relationship for any number of counters.

Consolidation

(10 to 15 min)

Apply the Math (Pairs, Class), pages 53 to 54

Using the Solved Examples

In Example 1, students use inductive reasoning to determine possible solutions for a winning turn. The analysis of winning moves opens up various strategies for the players.

In Example 2, students are given a model for analyzing a winning strategy by working backward from a win.

Sample Answers to Your Turn Questions

Example 1: a) Frank could also win by hitting two double 9s \([(2)(9) + (2)(9) = 36]\) or by hitting 20 followed by a double 8 \([20 + (2)(8) = 36]\).

b) Tara needs 100 to win. She could score 100 by hitting a triple 20 followed by a double 20 \([(3)(20) + (2)(20) = 100]\)

Example 2: a) The part of Alice’s strategy that involved deductive reasoning was the first step. Alice determined the only possible situation in which she could guarantee that she would win. Then she continued to work backward, identifying the other guaranteed winning steps she needed to make.

b) The part of Alice’s strategy that involved inductive reasoning was recognizing the pattern and accepting that the logic developed in the previous steps would work for the extended pattern. Using the extension of a pattern simply because it is a pattern is evidence of inductive reasoning.

Sample Solution to the Key Question

7. a) There are three different solutions (or six, or twenty-four, depending on how students think about the permutations).
b) I notice that all the solutions have an odd number in the vertex position of the V shape. I also notice a pattern in the arms of the V. Once an odd number has been chosen for the vertex, one arm has the high and low values of the remaining numbers, while the other arm has the two middle values of the remaining numbers.

c) To convince someone that I have identified all the possible solutions, I would have to prove that a solution with an even number at the vertex is not possible, and that there were no other solutions with an odd number at the vertex.

Closing (Groups, Class), page 57

For question 14, half the students could be asked to explain how inductive reasoning can help them develop a strategy to play a game or solve a puzzle, while the other half could be asked to explain how deductive reasoning can help. When both groups have finished, they can present their explanation to the other half of the class. The presentations could be set up as a debate or as advertising. Ensure that sufficient time is provided for students to complete their presentations for both inductive and deductive reasoning.

<table>
<thead>
<tr>
<th>Assessment and Differentiating Instruction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>What You Will See Students Doing...</strong></td>
</tr>
<tr>
<td><strong>When students understand...</strong></td>
</tr>
<tr>
<td>Students play the games in an analytical frame of mind, looking for patterns and thinking of strategies.</td>
</tr>
<tr>
<td>Students identify when they are using inductive and deductive reasoning to solve problems.</td>
</tr>
<tr>
<td><strong>If students misunderstand...</strong></td>
</tr>
<tr>
<td>Students may play the games and solve the problems, but they may not be able to determine or articulate a winning strategy.</td>
</tr>
<tr>
<td>Students are unable to differentiate between inductive and deductive reasoning, including when to apply each.</td>
</tr>
<tr>
<td><strong>Key Question 7</strong></td>
</tr>
<tr>
<td>Students consider the patterns within the set of possible values.</td>
</tr>
<tr>
<td>Students explain their strategy using math language.</td>
</tr>
<tr>
<td>**Differentiating Instruction</td>
</tr>
<tr>
<td><strong>EXTRA SUPPORT</strong></td>
</tr>
<tr>
<td>1. If students have difficulty with the problems and games, simpler ones may be substituted. For example, easier Sudoku puzzles are available online or in puzzle books.</td>
</tr>
<tr>
<td>2. Students may find it helpful to articulate their strategy by creating an <em>if … then</em> list with explanations. For example, “If I do this, then what would you do? Why? If I did this instead, then what would you do? Why?”</td>
</tr>
</tbody>
</table>
Using the Chapter Self-Test

Encourage students to use the Chapter Self-Test for self-assessment. Working on the questions should help them identify areas of need. If students are unable to complete a question, refer them to the relevant Frequently Asked Questions on Student Book pages 34, 59, and 60 and to the examples listed in the Study Aid next to these questions. Once students have reviewed the examples and understand the solutions and explanations, they can attempt the Chapter Review questions listed in the Study Aid beside the Frequently Asked Questions.

**What Do You Think Now? (Individual, Class), page 58**
Revisit What Do You Think? for Chapter 1 on Student Book page 5. Have students look back at their initial decisions and/or explanations. Students can compare their ideas then and now, and reflect on what they have learned. Students should be able to give correct and complete answers by the end of the chapter.

1. **Disagree.** Answers may vary, e.g., a pattern over a short time may not be true all the time. Four people exiting a shop with coffee cups in their hands does not mean that the next person leaving the shop will be holding a coffee cup.

2. **Agree.** Answers may vary, e.g., a pattern may be seen after examining several examples. After four people exit a shop with coffee cups, a prediction can be made that the shop sells coffee. However, more examples are needed.

3. **Disagree.** Answers may vary, e.g., the pattern can be described as increasing squares but it can also be described as the sum of the preceding number and the next odd number: \(0 + 1, 1 + 3, 4 + 5, 9 + 7, 16 + 9\). In both descriptions, the next three terms are 36, 49, and 64.
CHAPTER REVIEW

Using the Frequently Asked Questions
Have students keep their Student Books closed while you display the Frequently Asked Questions without the answers. Discuss the questions as a class, and use the discussion to draw out what students think is a good answer to each question. Then have students compare the class answers with the answers on Student Book pages 59 and 60. Remind students to refer to the answers in the Student Book as they work through the Practising questions and when they review later.

Using the Chapter Review
Ask students if they have questions or wish to share any insights about the topics covered in the chapter. Review the topics that students would benefit from considering again. Assign Practising questions as needed.

To gain greater insight into students’ understanding of the topics covered in the chapter, ask questions such as these:
- How do inductive and deductive reasoning differ?
- How might both types of reasoning be used to solve a problem?
- Why might a conjecture be revised?

<table>
<thead>
<tr>
<th>Question</th>
<th>Curriculum</th>
<th>Processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LR1.1</td>
<td>CN, PS, R</td>
</tr>
<tr>
<td>2</td>
<td>LR1.1</td>
<td>C, CN, R</td>
</tr>
<tr>
<td>3</td>
<td>LR1.1</td>
<td>CN, R</td>
</tr>
<tr>
<td>4</td>
<td>LR1.1, LR1.4</td>
<td>CN, CN, PS, R</td>
</tr>
<tr>
<td>5</td>
<td>LR1.4</td>
<td>C</td>
</tr>
<tr>
<td>6</td>
<td>LR1.4, LR1.7</td>
<td>C, R</td>
</tr>
<tr>
<td>7</td>
<td>LR1.4, LR1.7</td>
<td>C, R</td>
</tr>
<tr>
<td>8</td>
<td>LR1.1</td>
<td>C, CN, R</td>
</tr>
<tr>
<td>9</td>
<td>LR1.6</td>
<td>C, PS, R</td>
</tr>
<tr>
<td>10</td>
<td>LR1.5</td>
<td>C</td>
</tr>
<tr>
<td>11</td>
<td>LR1.1, LR1.6</td>
<td>C, PS, R</td>
</tr>
<tr>
<td>12</td>
<td>LR1.8</td>
<td>CN, PS</td>
</tr>
<tr>
<td>13</td>
<td>LR1.5, LR1.8</td>
<td>PS</td>
</tr>
<tr>
<td>14</td>
<td>LR2.1</td>
<td>PS, R</td>
</tr>
<tr>
<td>15</td>
<td>LR2.1</td>
<td>PS, R</td>
</tr>
<tr>
<td>16</td>
<td>LR2.1</td>
<td>C, PS, R</td>
</tr>
<tr>
<td>17</td>
<td>LR2.1</td>
<td>PS, R</td>
</tr>
</tbody>
</table>
CHAPTER TASK
How Many Sisters and Brothers?

WNCP
Specific Outcome
Logical Reasoning
1. Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. [C, CN, PS, R]

Achievement Indicators
1.1 Make conjectures by observing patterns and identifying properties, and justify the reasoning.
1.4 Provide and explain a counterexample to disprove a given conjecture.
1.5 Prove algebraic and number relationships, such as divisibility rules, number properties, mental mathematics strategies or algebraic number tricks.
1.6 Prove a conjecture, using deductive reasoning (not limited to two column proofs).

Introducing the Chapter Task (Class)
During the unit, students have had several encounters with number tricks. Review these encounters with students. They have been asked to make conjectures about the relationship between the starting number and resulting number (Lesson 1.3), to find a counterexample to the conjecture for a number trick (Lesson 1.3), to develop a proof for a number trick (Lesson 1.4), and to identify the error in a given proof (Lesson 1.5). In this Chapter Task, students are asked to develop a number trick of their own.

Examine the samples of number tricks developed by Rob, Wynn, and Yu. Review the criteria, considering the samples given. Note that Rob, Wynn, and Yu have only partially completed the first prompt. Discuss what a completed Chapter Task would include and what each piece would look like.

Using the Chapter Task
Have students work individually on the first step of the task. Then have them trade number tricks with a classmate to complete the task, still working individually. Remind students to include their classmate’s number trick with their completed task. Also remind them to review both the criteria for the prompts and the Task Checklist before submitting their task.

Sample Solution to the Chapter Task
A. Answers may vary, e.g.,
My number trick: Choose a number. Square it. Subtract 12. Multiply by 0. Add 5.
Test:

<table>
<thead>
<tr>
<th>Choose a number</th>
<th>Square it</th>
<th>Subtract 12</th>
<th>Multiply by 0</th>
<th>Add 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>144</td>
<td>132</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-8</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>-10</td>
<td>100</td>
<td>88</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

B. Answers may vary, e.g.,

<table>
<thead>
<tr>
<th>Choose a number</th>
<th>Add 2</th>
<th>Multiply by 5</th>
<th>Subtract 5</th>
<th>Divide by 5</th>
<th>Multiply by 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>25</td>
<td>20</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>60</td>
<td>55</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>10</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

C. My conjecture is that my classmate has no siblings.

D. Proof of conjecture:

<table>
<thead>
<tr>
<th>Choose a number</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add 2.</td>
<td>( m + 2 )</td>
</tr>
<tr>
<td>Multiply by 5.</td>
<td>( 5m + 10 )</td>
</tr>
<tr>
<td>Subtract 5.</td>
<td>( 5m + 5 )</td>
</tr>
<tr>
<td>Divide by 5.</td>
<td>( m + 1 )</td>
</tr>
<tr>
<td>Multiply by 0.</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ (m + 2)5 = 5m + 10 \]
\[ 5m + 10 - 5 = 5m + 5 \]
\[ \frac{(5m + 5)}{5} = m + 1 \]

E. No. Since the last step involves multiplying by zero, the answer will always be zero, no matter what number you start with.

Assessing Students’ Work
Use the Assessment of Learning chart as a guide for assessing students’ work.

Adapting the Task
You can adapt the task in the Student Book to suit the needs of your students. For example:

- Have students work in pairs so that stronger students can help students who are experiencing difficulty.
- Have weaker students include fewer steps in their number trick. Instead of proving the conjecture, ask students to change one step in the classmate’s number trick and then gather further evidence of how this change would affect the result.
- Have stronger students use a different arithmetic operation in each step of their number trick. Ask students to prove their own number trick, as well as their classmate’s number trick.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prompt A</strong> (Communication, Problem Solving) <strong>WNCP LR1</strong></td>
<td>Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems.</td>
<td>* uses effective and specific mathematical language and conventions to enhance communication (e.g., using language for the number trick)</td>
<td>* uses appropriate and correct mathematical language and conventions to support communication (e.g., using language for the number trick)</td>
<td>* uses mathematical and non-mathematical language and conventions incorrectly and/or inconsistently, interfering with communication (e.g., using language for the number trick)</td>
</tr>
<tr>
<td></td>
<td>• uses effective and specific mathematical language and conventions to enhance communication (e.g., using language for the number trick)</td>
<td>• shows flexibility and insight when solving the problem, adapting if necessary (e.g., creating a number trick)</td>
<td>• shows thoughtfulness when solving the problem (e.g., creating a number trick)</td>
<td>• attempts to verify the solution and determine the appropriateness of the response, sometimes incorrectly; draws basic conclusions based on sufficient evidence (e.g., testing the number trick)</td>
</tr>
<tr>
<td></td>
<td>• verifies the solution and accurately determines the appropriateness of the response; draws insightful conclusions based on all available evidence (e.g., testing the number trick)</td>
<td>• verifies the solution and reasonably determines the appropriateness of the response; draws appropriate conclusions based on relevant evidence (e.g., testing the number trick)</td>
<td>• verifies the solution and reasonably determines the appropriateness of the response; draws appropriate conclusions based on relevant evidence (e.g., testing the number trick)</td>
<td></td>
</tr>
<tr>
<td><strong>Prompts B and C</strong> (Connections) <strong>WNCP LR1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>* demonstrates a sophisticated ability to transfer knowledge and skills to new contexts (e.g., relating the pattern of operations to the result)</td>
<td>* demonstrates a consistent ability to transfer knowledge and skills to new contexts (e.g., relating the pattern of operations to the result)</td>
<td>* demonstrates some ability to transfer knowledge and skills to new contexts (e.g., relating the pattern of operations to the result)</td>
<td>* demonstrates a limited ability to transfer knowledge and skills to new contexts (e.g., relating the pattern of operations to the result)</td>
</tr>
<tr>
<td></td>
<td>* shows flexibility and insight when solving the problem, adapting if necessary (e.g., proving the conjecture deductively)</td>
<td>* shows thoughtfulness when solving the problem (e.g., proving the conjecture deductively)</td>
<td>* shows understanding when solving the problem (e.g., proving the conjecture deductively)</td>
<td>* attempts to solve the problem (e.g., proving the conjecture deductively)</td>
</tr>
</tbody>
</table>
### Assessment of Learning—What to Look for in Student Work...

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Prompt E</td>
<td>• comprehensively analyzes situations and makes insightful generalizations (e.g., finding a counterexample to the number trick)</td>
<td>• completely analyzes situations and makes logical generalizations (e.g., finding a counterexample to the number trick)</td>
<td>• superficially analyzes situations and makes simple generalizations (e.g., finding a counterexample to the number trick)</td>
<td>• is unable to analyze situations and make generalizations (e.g., finding a counterexample to the number trick)</td>
</tr>
<tr>
<td>WNCP LR1</td>
<td>• provides a precise and insightful explanation of mathematical concepts and/or procedures (e.g., using mathematical terminology within the justification)</td>
<td>• provides a clear and logical explanation of mathematical concepts and/or procedures (e.g., using mathematical terminology within the justification)</td>
<td>• provides a partially clear explanation of mathematical concepts and/or procedures (e.g., using mathematical terminology within the justification)</td>
<td>• provides a vague and/or inaccurate explanation of mathematical concepts and/or procedures (e.g., using mathematical terminology within the justification)</td>
</tr>
</tbody>
</table>
PROJECT CONNECTION 1
Creating an Action Plan

WNCP
Specific Outcome
Mathematics Research Project
1. Research and give a presentation on a current event or an area of interest that involves mathematics. [C, CN, ME, PS, R, T, V]

Achievement Indicator
1.1 Collect primary or secondary data (statistical or informational) related to the topic.

Using the Project Connection
At the end of each chapter, support is provided in the Project Connection to help students meet the project outcome. Use the Project Connection to help students manage the process, to ensure that students’ project work is ongoing by refocusing their attention, and to support students so they see the project as a step-by-step process and are not overwhelmed by the idea of a big project. Help students understand that if they break the project work into chunks, it becomes manageable.

Specific to Chapter 1 is the important step of preparing a tentative schedule. This will prompt students to think ahead about what needs to be done, while keeping the project work in discrete chunks. As well, this will make it easier for you to help each student manage the project work.

You might begin by allowing students time to read Creating an Action Plan and Issues Affecting Project Completion on page 64. Then brainstorm, as a class, other issues that could be added to the list. You might even discuss possible strategies for dealing with these issues. A project completion date should be set at this time (because students need to work backward from the completion date in step B of Your Turn to create their schedule).

Students can work in pairs or small groups to complete the steps in Your Turn. There is a blackline master available for this, if you wish to use it. Students can think back to previous projects they have worked on when estimating their timing at each stage. As students work, you could have selected students share with the class some of their estimates for the timing of each stage and explain their reasoning.

Some students may have difficulty creating a schedule without knowing what their topic will be. Tell students that they will be choosing their topic at the end of Chapter 2, but they can work with a tentative topic at this point if it helps them create their schedule.

It is very important to emphasize that this schedule will be revisited on an ongoing basis throughout the course. For example, at the end of Chapter 2, students will examine their schedule and possibly make adjustments based on their chosen topic.
CHAPTER 1 DIAGNOSTIC TEST

STUDENT BOOK PAGES 2–63

1. Determine one or more values for each variable in the Venn diagram.

2. Complete the patterns.
   a) 1, 4, ___, ___, 25, ___, ___, ___, 100, ___, ___
   b) 2, 4, 8, 16, ___, ___, ___, 256

3. Identify three prime numbers that are less than 20.

4. Identify which number is not a multiple of 24.
   48  94 120  144

5. Determine the value of each expression.
   a) $2^4$
   b) $4^2 - 3^2$
   c) $\sqrt{64}$
   d) $(-5)(3)$
   e) $(5 + 8)(-2) - (-12)$

6. Represent each phrase as an algebraic expression.
   a) a number squared
   b) the sum of two consecutive numbers
   c) the product of half a number and 5
   d) the difference between twice a number and 7

7. Factor each expression.
   a) $4x - 12$
   b) $6x + 10$
   c) $x^2 - 16$
   d) $24 + 8m + 16t$
   e) $5b - xb$
   f) $r^2 + 2r - 3$

8. Evaluate the expressions in question 7, parts a), b), and c), for $x = 10$.

   a) $5(2x + 3c)$
   b) $(3x + y)(x - 2y)$

10. Solve each equation.
    a) $3x - 5 = 10$
    b) $8 + 2y = 11$

11. Draw a parallelogram with side lengths of 3 cm and 7 cm.
    a) Bisect the sides of the parallelogram, and join the adjacent midpoints.
    b) Use a different colour to draw the diagonals of the parallelogram.

12. Draw a scalene triangle, and construct its medians.
13. Examine the following pattern. Predict the next product.

\[
\begin{align*}
4 (4) &= 16 \\
4 (44) &= 176 \\
4 (444) &= 1776 \\
4 (4444) &= 17776 \\
4 (44444) &= 177776
\end{align*}
\]

14. For each set of intersecting lines, identify vertically opposite angles and supplementary angles.

a) 

```
  a
  \\
  d
  b
  c
```

b) 

```
  q
  \\
  r
  t
  s
```
CHAPTER 1 DIAGNOSTIC TEST ANSWERS

1. Answers may vary, e.g., \(a = 12, b = 20, c = 33, d = 23\)

2. a) 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144
   b) 2, 4, 8, 16, 32, 64, 128, 256

3. Answers may vary. Any three of 2, 3, 5, 7, 11, 13, 17, 19.

4. 94

5. a) 16  b) 7  c) 8 or 8  d) −15  e) −14

6. Answers may vary, e.g.,
   a) \(n^2\)  b) \((n) + (n + 1)\)  c) \(\left(\frac{n}{2}\right)^5\)  d) \(2n - 7\)

7. a) \(4(x - 3)\)  d) \(8(3 + m + 2t)\)
   b) \(2(3x + 5)\)  e) \(b(5 - x)\)
   c) \((x + 4)(x - 4)\)  f) \((r - 1)(r + 3)\)

8. a) 28  b) 70  c) 84

9. a) \(10x + 15c\)  b) \(3x^2 - 5xy - 2y^2\)

10. a) \(x = 5\)  b) \(v = 1.5\)

11. Answers may vary, e.g.,
   a) 
   b) 

12. Answers may vary, e.g.,

13. Answers may vary, e.g., the next product will start with the digit 1 and end with the digit 6. The only other digit will be 7.

14. a) Vertically opposite angle pairs: \(a\) and \(c\), \(b\) and \(d\); supplementary angle pairs: \(a\) and \(b\), \(b\) and \(c\), \(c\) and \(d\), \(d\) and \(a\)
   b) Vertically opposite angle pairs: \(q\) and \(s\), \(r\) and \(t\); supplementary angle pairs: \(q\) and \(r\), \(r\) and \(s\), \(s\) and \(t\), \(t\) and \(q\)

If students have difficulty with questions in the Diagnostic Test, you may need to review the following topics:

- Classifying numbers
- Factoring, evaluating, and solving algebraic expressions and equations
- Drawing specific geometric figures
REVIEW OF TERMS AND CONNECTIONS

WORDS You Need to Communicate Effectively
1. Match each term with a diagram or example below.
   a) three-digit number  
   b) congruent shapes  
   c) equivalent form  
   d) expanded form  
   e) perfect square  
   f) prime number  
   g) supplementary angles  
   h) vertically opposite angles

   i) 81  ii) 135  iii) 100

   iv) 61  v) 148 = 100 + 40 + 8  vi) 110

   vii) 4d = 3d + 1d  
   viii) 8

CONNECTIONS You Need for Success
Working with Algebraic Expressions and Equations
Algebraic expressions may be represented symbolically with words or with variables, coefficients, and constants. Through the processes of simplification, expansion, and evaluation, algebraic expressions can be flexibly manipulated. For example:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 2)(x - 1) - x^2 + 5)</td>
<td>Given expression</td>
</tr>
<tr>
<td>((x^2 + 2x - 1x - 2) - x^2 + 5)</td>
<td>Expand the expression.</td>
</tr>
<tr>
<td>(x^2 + x - 2 - x^2 + 5)</td>
<td>Gather like x terms.</td>
</tr>
<tr>
<td>(x - 2 + 5)</td>
<td>Gather like x² terms.</td>
</tr>
<tr>
<td>(x + 3)</td>
<td>Gather constants.</td>
</tr>
</tbody>
</table>

The simplified expression, \(x + 3\), is much easier to evaluate than \((x + 2)(x - 1) - x^2 + 5\).

2. Simplify each expression. Then evaluate for \(a = 5\), \(d = 6\), and \(v = 7\).
   a) \(a^2 - 5a + 6a + 3a^2\)
   b) \(17 - d + 2(3d + 2)\)
   c) \((v + 2)(v - 2) + 13\)
When you are asked to solve an algebraic equation, you need to determine the value of an unknown. Solving an algebraic equation requires a systematic approach, as well as an understanding of how to manipulate algebraic expressions (as shown above). You should also understand inverse operations and the order of operations. For example:

<table>
<thead>
<tr>
<th>Step</th>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2(3x + 5) = x$</td>
<td>Solve for $x$ in the given equation.</td>
</tr>
<tr>
<td>2</td>
<td>$6x + 10 = x$</td>
<td>Expand the left side of the equation.</td>
</tr>
<tr>
<td>3</td>
<td>$6x + 10 - x = x - x$</td>
<td>Subtract $x$ from each side to begin isolating the variable $x$ on one side of the equation.</td>
</tr>
<tr>
<td>4</td>
<td>$5x + 10 - 10 = 0 - 10$</td>
<td>Subtract 10 from each side.</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5x}{5} = \frac{-10}{5}$</td>
<td>Divide each side by 5 to isolate $x$ and solve the equation.</td>
</tr>
</tbody>
</table>

3. Solve each equation.
   a) $2x + 4 = 10$
   b) $\frac{x}{2} - 3 = 5$
   c) $2x^2 = 32$

**Applying Number Concepts**

**Number Classification**

Sometimes problems identify a set of numbers to which an unknown value belongs. Knowing what is included in the set of numbers allows you to choose an appropriate value. For example:

- Double a natural number means 2 times a counting number such as 3, 7, or 15.
- Half a rational number means 0.5 times a number such as $-\frac{5}{8}$, 1.36, or even 0.

4. Identify a number that matches each description.
   a) a whole number that is not a natural number
   b) an integer that is not a whole number
   c) a rational number that is not an integer

**Number Properties**

Within the set of whole numbers, there are special types of numbers. For example:

- A factor is a number that divides exactly into another number: 6 is a factor of 12 because $12 \div 6 = 2$.
- A multiple is the product of a number and a whole number: 12 is a multiple of 4 because $3 \times 4 = 12$.
- A prime number, such as 7, has exactly two factors, 1 and the number itself: $7 = 7 \times 1$.
- A perfect square, such as 81, can be named as the product of a number with itself: $81 = 9 \times 9$.

5. a) Show why 144 can be a multiple, a factor, and a perfect square.
   b) Show why 144 cannot be a prime number.
PRACTISING

6. Order these expressions from least to greatest.
   a) \(2^3 - 5^2\)
   b) the difference between \(4^2\) and 3
   c) the sum of the first three positive odd numbers
   d) the product of \(-4\) and 3

7. Provide one or more numbers to match each description. For example, the number for part a) could be 20 because \(4 \times 5 = 20\) and 5 is a whole number.
   a) four times a whole number
   b) the sum of two consecutive integers
   c) a multiple of 7 less than 50
   d) a factor of 24 that is odd
   e) the square root of an even number
   f) a sum of two perfect squares that is greater than 50

8. Factor or expand each expression. Then evaluate for \(x = 2\).
   a) \(4x^3 - 40\)
   b) \(5x^2 + 20x - 3\)
   c) \((x + 3)(2x - 5)\)
   d) \(7x(3x^3 + 5x)\)

9. Which equations have a solution of \(x = 2\)?
   a) \(\frac{1}{2}x + 3 = 4\)
   b) \(6x^2 - 3 = 21\)
   c) \(\frac{x}{10} = 0.5\)

10. Determine angles \(a\), \(b\), and \(c\) without measuring.

11. Sketch each shape. Show all the congruent side lengths and right angles.
    a) a rectangle that is not a square
    b) a rhombus that is not a square
    c) a quadrilateral that is not a right trapezoid

12. Examine the pattern below. Make a prediction about the next number.
    \[2^2 = 4\]
    \[22^2 = 484\]
    \[222^2 = 49284\]
    \[2222^2 = 4937284\]
    \[222222^2 = 493817284\]
REVIEW OF TERMS AND CONNECTIONS ANSWERS

1. a) ii) e) i)
b) viii) f) iv)
c) vii) g) vii)
d) v) h) iii)

2. a) $4a^2 + a; 105$ b) $5d + 21; 51$ c) $v^2 + 9; 58$

3. a) $x = 3$ b) $x = 16$ c) $x = 4, -4$

4. a) 0
   b) Answers may vary, e.g., $-7$
c) Answers may vary, e.g., $-\frac{7}{8}$

5. Answers may vary, e.g.,
a) $144 = 3 \times 48; 288 \div 144 = 2; 144 = 12 \times 12$
b) $144 = 2 \times 72$

6. a), d), c), b)

7. Answers may vary, e.g.,
a) 40 ($4 \times 10 = 40$)
b) 25 ($12 + 13 = 25$)
c) 42 ($6(7) < 50$)
d) 3 ($24 \div 3 = 8$)
e) 4 ($16 = 4, -4$)
f) 61 ($5^2 + 6^2 = 61$)

8. a) $4(x^3 - 10); -8$
b) $5x(x + 4) - 3; 57$
c) $2x^2 + x - 15; -5$
d) $21x^4 + 35x^2; 476$

9. a) and b)

10. $a = 150^\circ, b = 30^\circ, c = 150^\circ$

11. Answers may vary, e.g.,

   a)

   b)

   c)

12. Answers may vary, e.g., the next number will have 11 digits. It will be 49 38_ _17 284.
EXPLORE THE MATH: OPTICAL ILLUSIONS

Seeing is believing, but eyes can be deceived.

Choose two of these four optical illusions.

Make a conjecture about diagonal $AB$ and diagonal $BC$.

Make a conjecture about the circles in the centre.

Make a conjecture about the lines.

Make a conjecture about the number of triangles.

How can you check the validity of your conjectures?
A. Solve the three triangular arithmagons below.

a) 

```
   17  18
  /    /
19
```

b) 

```
   9  12
  /    /
15
```

c) 

```
   38  48
  /    /
68
```
SOLVING PUZZLES (QUESTIONS 10 TO 13)

10. a) 

```
5  2  6
  4
1  6
  1
3  2  1
  6
```

b) 

```
6  4  8  2
  1  4
  6  3  5
  1
8  9  2
  1
4  5  6
  9  3
```

11. a) 

```
6
1
4  3  8
```

b) 

```
4
5  3
8
```

12. 

```
30 \times
36 \times
2 \div
18 +
3 +
7 +
20 \times
5 -
1 -
2 -
13 +
7 +
2 -
2 \div
3 -
```

13. 

```
36 \times
2 \div
18 +
7 +
5 -
13 +
2 -
3 -
```
Creating an Action Plan

A. Start by deciding on the probable length of time for each stage.

1. **Select the topic you would like to explore.**
   - Suggested time: 1 to 3 days
   - Your probable time: _____ Finish date: ________

2. **Create the research question you would like to answer.**
   - Suggested time: 1 to 3 days
   - Your probable time: _____ Finish date: ________

3. **Collect the data.**
   - Suggested time: 5 to 10 days
   - Your probable time: _____ Finish date: ________

   **Buffer space**
   - Suggested time: 3 to 7 days
   - Your probable time: _____ Finish date: ________

4. **Analyze the data.**
   - Suggested time: 5 to 10 days
   - Your probable time: _____ Finish date: ________

5. **Create an outline for your presentation.**
   - Suggested time: 2 to 4 days
   - Your probable time: _____ Finish date: ________

6. **Prepare a first draft.**
   - Suggested time: 3 to 10 days
   - Your probable time: _____ Finish date: ________

7. **Revise, edit, and proofread.**
   - Suggested time: 3 to 5 days
   - Your probable time: _____ Finish date: ________

8. **Prepare and practise your presentation.**
   - Suggested time: 3 to 5 days
   - Your probable time: _____ Finish date: ________

B. Use a calendar and your probable times for each stage to work backward from the presentation date to create a schedule you can follow. In your schedule, include regular conferences with your teacher—5 to 10 min to discuss your progress.
CHAPTER 1 TEST

1. Hilary was examining the differences between perfect squares of numbers separated by 5. She made the following conjecture: The differences always have the digit 5 in the ones place. For example: $17^2 - 12^2 = 289 - 144 = 145$
   a) Gather evidence to support Hilary’s conjecture.
   b) Is her conjecture reasonable? Explain.

2. Denyse works part time at a grocery store. She notices that the store is very busy when she works in the afternoon from 4 to 7 p.m., but it is less busy when she works in the evening from 7 to 10 p.m. What conjecture can you make for this situation? Justify your conjecture.

3. Heather claimed that the sum of two multiples of 4 is a multiple of 8. Is Heather’s conjecture reasonable? Explain. If it is not reasonable, find a counterexample.

4. Prove that the sum of two consecutive perfect squares is always an odd number.

5. Prove that the following number trick will always result in 6:
   - Choose any number.
   - Add 3.
   - Multiply by 2.
   - Add 6.
   - Divide by 2.
   - Subtract your starting number.

6. Judd presented the following argument:
   Inuvik, Northwest Territories, is above the Arctic Circle, which is at a latitude of 66° north of the equator. Places north of the Arctic Circle have cold, snowy winters. Winnipeg is at a latitude of 52° north of the equator. Therefore, Winnipeg does not have cold, snowy winters.
   Is Judd’s argument reasonable? If not, identify the errors in his reasoning.

7. Is this proof valid? Explain.

<table>
<thead>
<tr>
<th>Let any number, $a$, equal $b$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^2 = ab$</td>
</tr>
<tr>
<td>Multiply both sides by $a$.</td>
</tr>
<tr>
<td>$a^2 - b^2 = ab - b^2$</td>
</tr>
<tr>
<td>Subtract $b^2$ from both sides.</td>
</tr>
<tr>
<td>$(a + b)(a - b) = b(a - b)$</td>
</tr>
<tr>
<td>Factor both sides.</td>
</tr>
<tr>
<td>$a + b = b$</td>
</tr>
<tr>
<td>Divide by $(a - b)$.</td>
</tr>
<tr>
<td>$a + b - b = b - b$</td>
</tr>
<tr>
<td>Subtract $b$ from both sides.</td>
</tr>
<tr>
<td>$a = 0$</td>
</tr>
<tr>
<td>Any number equals zero.</td>
</tr>
</tbody>
</table>
CHAPTER 1 TEST ANSWERS

1. Answers may vary, e.g.,
   a) \(6^2 - 1^2 = 36 - 1 = 35\)
      \(7^2 - 2^2 = 49 - 4 = 45\)
      \(15^2 - 10^2 = 225 - 100 = 125\)
      \((-3)^2 - 2^2 = 9 - 4 = 5\)
   b) Hilary’s conjecture is reasonable since all the evidence supports it.
      Each difference ends with the digit 5 in the ones place.

2. Possible conjecture: More people shop on their way home from work than later in the evening. This conjecture is reasonable because more people get off work between 4 and 7 p.m. than between 7 and 10 p.m. The increased number of shoppers that Denyse notices between 4 and 7 p.m. could be people stopping on their way home from work.

3. No. Heather’s conjecture is not reasonable because it doesn’t always work. For example, the sum of 8 and 24 is a multiple of 8. The sum of 12 and 20 is a multiple of 8. But the sum of 8 and 12 is not a multiple of 8. This is a counterexample to the conjecture.

4. Answers may vary, e.g.,

<table>
<thead>
<tr>
<th>Let (x) and (x + 1) be two consecutive numbers.</th>
<th>Given</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 + (x + 1)^2)</td>
<td>Write an expression for the sum of two consecutive perfect squares.</td>
</tr>
<tr>
<td>(x^2 + x^2 + 1x + 1x + 1)</td>
<td>Simplify the expression.</td>
</tr>
<tr>
<td>(2x^2 + 2x + 1)</td>
<td>Combine like terms.</td>
</tr>
<tr>
<td>(2(x^2 + x) + 1)</td>
<td>Factor.</td>
</tr>
<tr>
<td>(2(x^2 + x))</td>
<td>Twice any number is even.</td>
</tr>
<tr>
<td>(2(x^2 + x) + 1)</td>
<td>One more than any even number is odd.</td>
</tr>
</tbody>
</table>

5. Choose any number.  \(d\)
   Add 3. \(d + 3\)
   Multiply by 2. \(2(d + 3) = 2d + 6\)
   Add 6. \(2d + 12\) \([2d + 6 + 6 = 2d + 12]\)
   Divide by 2. \(\frac{2d + 12}{2} = d + 6\)
   Subtract your starting number. \(6\) \([d + 6 - d = 6]\)

6. Judd’s argument is not reasonable because the second statement cannot be interpreted as \textit{only} places above the Arctic Circle have cold, snowy winters. Since his interpretation is false, his conclusion is invalid.

7. No, the proof is not valid because \(a - b\) equals zero and division by zero is undefined.