LEARNING GOALS

You will be able to develop algebraic and graphical reasoning by

- Modelling and solving problems algebraically and graphically using linear inequalities in two variables
- Modelling and solving problems algebraically and graphically using systems of linear inequalities in two variables
- Solving optimization problems using linear programming

How can linear inequalities help shoppers decide what and how much to buy?
Balancing Two Part-Time Jobs

Marnie has two part-time jobs. She earns minimum wage working at the information desk in a hospital and $4 above minimum wage helping with her mother’s house-cleaning business. Marnie works in whole-hour increments only. She enjoys the work at the hospital more than house cleaning and does not work more than 15 h a week, since she often has a lot of homework and she plays on her school’s volleyball team.

A. How many hours can Marnie work at the hospital and earn at least $160 a week?
A. Predict the solution to the problem. Explain your prediction to a partner.
B. If Marnie works 5 h at the hospital, how many hours will she work for her mother? Will she earn at least $160? Explain.
C. If Marnie works 10 h at the hospital, how many hours will she work for her mother? Will she earn at least $160? Explain.
D. What type of linear inequality is being described in the problem? Justify your choice.
E. Use a variable to represent the number of hours that Marnie works at the hospital. Use the same variable to create an expression for the number of hours that she works for her mother.
F. To what set of numbers—natural, whole, integer, rational, or real—does the variable you defined in part E belong? Justify your decision.

G. Write a linear inequality to represent Marnie’s situation. Explain why your linear inequality makes sense.

H. Solve the linear inequality, and represent the solution on a number line.

I. Describe how your prediction in part A compares with your solution in part H.

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. The solution to a linear inequality can be modelled graphically on a number line.
2. A linear inequality can be modelled using a solid line or dashed line.
3. The same algebraic methods that are used to solve a linear equation can be used to solve a linear inequality.
4. A linear equation has been graphed below. Its solutions can only be represented by points with integer coordinates, such as (1, 5) and (−1, 1).
Chapter 6  Systems of Linear Inequalities

Graphing Linear Inequalities in Two Variables

**INVESTIGATE the Math**

Amir owns a health-food store. He is making a mixture of nuts and raisins to sell in bulk. His supplier charges $25/kg for nuts and $8/kg for raisins.

What quantities of nuts and raisins can Amir mix together if he wants to spend less than $200 to make the mixture?

A. Suppose that Amir wants to spend exactly $200 to make the mixture. Work with a partner to create an equation that represents this situation.

B. To what set of numbers does the domain and range of the two variables in your equation belong? Use this information to help you graph the equation on a coordinate plane.

C. Explain why the graph is a line segment, not a ray or a line.

D. What region of the coordinate plane includes points representing quantities of nuts and raisins that Amir could use if he wants to spend less than $200? How do you know?

E. There are many possible solutions to Amir’s problem. Plot at least three points that represent reasonable solutions to Amir’s problem. Explain why you chose these points.

Reflecting

F. Discuss and then decide whether the solution set for Amir’s problem is represented by
   i) points in the region above the line segment.
   ii) points in the region below the line segment.
   iii) points on the line segment.

G. Why might the line segment be considered a boundary of the solution set?

H. Why might you use a dashed line segment for this graph instead of a solid line segment?
6.1 Graphing Linear Inequalities in Two Variables

I knew that the graph of the linear equation
\[ 2x + 5y = 10 \]
would form the boundary of the linear inequality
\[ 2x + 5y \geq 10. \]

The domain and range are not stated and no context is given, so I assumed that the domain and range are the set of real numbers. This means that the solution set is continuous.

A connected set of numbers. In a continuous set, there is always another number between any two given numbers. Continuous variables represent things that can be measured, such as time.

I knew that I needed to plot and join only two points to graph the linear equation. I decided to plot the two intercepts.

To determine the \( y \)-intercept, I substituted 0 for \( x \).

\[ 2x + 5y = 10 \]
\[ 2(0) + 5y = 10 \]
\[ 5y = 10 \]
\[ y = 2 \]
The \( y \)-intercept is at \((0, 2)\).

To determine the \( x \)-intercept, I substituted 0 for \( y \).

\[ 2x + 5y = 10 \]
\[ 2x + 5(0) = 10 \]
\[ 2x = 10 \]
\[ x = 5 \]
The \( x \)-intercept is at \((-5, 0)\).
Since my test point below the boundary was not a solution, I shaded the half plane that did not include my test point. This was the region above the boundary.}

Communication Tip
If the solution set to a linear inequality is continuous and the sign includes equality ($\leq$ or $\geq$), a solid green line is used for the boundary, and the solution region is shaded green, as shown to the right.

Since my test point below the boundary was not a solution, I shaded the half plane that did not include my test point. This was the region above the boundary.

I used green shading to show that the solution set belongs to the set of real numbers.

Since the domain and range are in the set of real numbers, I knew that the solution set is continuous. Therefore, the solution region includes all points in the shaded area and on the solid boundary.
Janet’s Solution: Using graphing technology

\[-2x + 5y \geq 10\]
\[5y \geq 2x + 10\]
\[\frac{5y}{5} \geq \frac{2x + 10}{5}\]
\[y \geq \frac{2x}{5} + 2\]

The variables represent numbers from the set of real numbers.
\[x \in \mathbb{R} \text{ and } y \in \mathbb{R}\]

To enter the linear inequality into my graphing calculator, I had to isolate \(y\).
I didn’t have to reverse the inequality sign because I didn’t divide or multiply by a negative value.

The domain and range are not stated and no context is given, so I assumed the domain and range are in the set of real numbers. This means that the solution set is continuous.

I entered the inequality into the calculator and graphed it.
I knew that the boundary was correctly shown because a solid line means equality is possible.

To verify whether the correct half plane was shaded, I used \((1, 4)\) in the half plane above the boundary as a test point.
Since \((1, 4)\) is a solution to the linear inequality, I knew that the half plane above the boundary should be shaded.

The solution region includes all the points in the shaded area and along the boundary because the solution set is continuous.

Your Turn

Compare the graphs of the following relations. What do you notice?
\[-2x + 5y \geq 10\]
\[-2x + 5y = 10\]
\[-2x + 5y < 10\]
Graphing linear inequalities with vertical or horizontal boundaries

Graph the solution set for each linear inequality on a Cartesian plane.

a) \( \{(x, y) \mid x - 2 > 0, x \in \mathbb{R}, y \in \mathbb{R}\} \)

b) \( \{(x, y) \mid -3y + 6 \geq -6 + y, x \in \mathbb{I}, y \in \mathbb{I}\} \)

**Wynn’s Solution**

a) \( x - 2 > 0 \)

\[ x > 2 \]

The variables represent numbers from the set of real numbers.

\( x \in \mathbb{R} \) and \( y \in \mathbb{R} \)

I isolated \( x \) so I could graph the inequality.

The domain and range are stated as the set of real numbers. The solution set is continuous, so the solution region and its boundary will be green in my graph.

I drew the boundary of the linear inequality as a dashed green line because I knew that the linear inequality \( (>) \) does not include the possibility of \( x \) being equal to 2.

I needed to decide which half plane to shade. For \( x \) to be greater than 2, I knew that any point to the right of the boundary would work.

The solution region includes all the points in the shaded area because the solution set is continuous. The solution region does not include points on the boundary.

**Communication Tip**

If the solution set to a linear inequality is continuous and the sign does not include equality \( (\leq, \geq) \), a dashed green line is used for the boundary and the solution region is shaded green, as shown to the right.
b) \(-3y + 6 \geq -6 + y\)
\[-4y \geq -12\]
\[\frac{-4y}{-4} \leq \frac{-12}{-4}\]
\[y \leq 3\]

Since the linear inequality has only one variable, \(y\), I isolated the \(y\).
As I rearranged the linear inequality, I divided both sides by \(-4\). That’s why I reversed the sign from \(\geq\) to \(\leq\).

The domain and range are stated as being in the set of integers. I knew this means that the solution set is **discrete**.

I knew that points with integer coordinates below the line \(y = 3\) were solutions, so I shaded the half plane below it orange.

I knew the linear inequality includes \(3\), so points on the boundary with integer coordinates are also solutions to the linear inequality.

I stippled the boundary and the orange half plane with green points to show that the solution set is discrete.

The solution region includes only the points with integer coordinates in the shaded region and along the boundary.

**Your Turn**

How can you tell if the boundary of a linear inequality is vertical or horizontal without graphing the linear inequality? Explain.
A sports store has a net revenue of $100 on every pair of downhill skis sold and $120 on every snowboard sold. The manager’s goal is to have a net revenue of more than $600 a day from the sales of these two items. What combinations of ski and snowboard sales will meet or exceed this daily sales goal? Choose two combinations that make sense, and explain your choices.

**Jerry’s Solution**

The relationship between the number of pairs of skis, \( x \), the number of snowboards, \( y \), and the daily sales can be represented by the following linear inequality:

\[
100x + 120y > 600
\]

The variables represent whole numbers.

\( x \in \mathbb{W} \) and \( y \in \mathbb{W} \)

\[
100x + 120y > 600 \\
120y > 600 - 100x \\
\frac{120y}{120} > \frac{600 - 100x}{120} \\
y > \frac{600 - 100x}{120} \\
y > \frac{5x}{6} \\
y > -\frac{5}{6} + 5
\]

\[\{(x, y) | 100x + 120y > 600, x \in \mathbb{W}, y \in \mathbb{W}\}\]

I defined the variables in this situation and wrote a linear inequality to represent the problem.

I know that only whole numbers are possible for \( x \) and \( y \) since stores don’t sell parts of skis or snowboards.

Because the domain and range are restricted to the set of whole numbers, I knew that the solution set is discrete.

I also knew that my graph would occur only in the first quadrant.

I isolated \( y \) so I could enter the inequality into my graphing calculator.

I adjusted the calculator window to show only the first quadrant, since the domain and range are both the set of whole numbers.

The boundary is a dashed line, which means that the solution set does not include values on the line.
When I interpreted the graph, I considered the context of the problem. I knew that
- only discrete points with whole-number coordinates in the solution region made sense.
- points along the dashed boundary are not part of the solution region.
- points with whole-number coordinates along the x-axis and y-axis boundaries are part of the solution region.

I picked two points in the solution region, (4, 4) and (5, 3), as possible solutions to the problem. I verified that each point is a solution to the linear inequality.

Sales of four pairs of skis and four snowboards or sales of five pairs of skis and three snowboards will exceed the manager’s net revenue goal of more than $600 a day.

**Your Turn**

a) Would raising the daily sales goal to at least $1000 change the graph that models this situation? Explain.

b) State two combinations of ski and snowboard sales that would meet or exceed this new daily sales goal.
In Summary

Key Idea
• When a linear inequality in two variables is represented graphically, its boundary divides the Cartesian plane into two half planes. One of these half planes represents the solution set of the linear inequality, which may or may not include points on the boundary itself.

Need to Know
• To graph a linear inequality in two variables, follow these steps:
  Step 1. Graph the boundary of the solution region.
  • If the linear inequality includes the possibility of equality (≤ or ≥), and the solution set is continuous, draw a solid green line to show that all points on the boundary are included.
  • If the linear inequality includes the possibility of equality (≤ or ≥), and the solution set is discrete, stipple the boundary with green points.
  • If the linear inequality excludes the possibility of equality (< or >), draw a dashed line to show that the points on the boundary are not included.
    - Use a dashed green line for continuous solution sets.
    - Use a dashed orange line for discrete solution sets.
  Step 2. Choose a test point that is on one side of the boundary.
  • Substitute the coordinates of the test point into the linear inequality.
  • If possible, use the origin, (0, 0), to simplify your calculations.
  • If the test point is a solution to the linear inequality, shade the half plane that contains this point.
    Otherwise, shade the other half plane.
    - Use green shading for continuous solution sets.
    - Use orange shading with green stippling for discrete solution sets.

For example,
\[
\{ (x, y) \mid y \leq -2x + 5, \quad x \in \mathbb{R}, y \in \mathbb{R} \} \\
\{ (x, y) \mid y > -2x + 5, \quad x \in \mathbb{R}, y \in \mathbb{R} \} \\
\{ (x, y) \mid y \geq -2x + 5, \quad x \in \mathbb{I}, y \in \mathbb{I} \} \\
\{ (x, y) \mid y \leq -2x + 5, \quad x \in \mathbb{I}, y \in \mathbb{I} \}
\]

• When interpreting the solution region for a linear inequality, consider the restrictions on the domain and range of the variables.
  - If the solution set is continuous, all the points in the solution region are in the solution set.
  - If the solution set is discrete, only specific points in the solution region are in the solution set.
    This is represented graphically by stippling.
  - Some solution sets may be restricted to specific quadrants. For example, most linear inequalities representing real-world problem situations have graphs that are restricted to the first quadrant.
CHECK Your Understanding

1. Graph the solution set for each linear inequality.
   a) \( y < x + 4 \)  
   b) \(-y < -6x + 3\)

2. Consider the graph of this inequality.
   \( 2x + 3y > 5 \)
   Make each of the following decisions, and provide your reasoning.
   a) whether the boundary should be dashed, stippled, or solid
   b) whether the half plane above or below the boundary should be shaded
   c) whether each point is in its solution region:
      i) (1, 1)  
      ii) (1, 0)  
      iii) (1, 2)

3. Betsy and Flynn work at an ice cream stand. If Betsy worked three times as many hours as she usually does and Flynn worked twice the number of hours that he usually does, together they would work less than 25 h. The situation can be modelled by the following linear inequality:
   \( 3b + 2f < 25 \)
   a) What do the variables \( b \) and \( f \) represent?
   b) What restrictions does the context place on the variables? Explain.
   c) Suppose you were to graph the inequality:
      i) Describe the boundary.
      ii) Would you shade the half plane above or below the boundary?
      iii) Would your graph involve all four quadrants? Explain.
   d) What does a solution to this inequality represent?

PRACTISING

4. Match each graph with its linear inequality, and justify your match.
   a)
   \[ \{(x, y) | x - 3 > -y, x \in W, y \in W\} \]
   b)
   \[ \{(x, y) | x - y > -3, x \in R, y \in R\} \]
   c)
   \[ \{(x, y) | y - 3 \geq x, x \in R, y \in R\} \]

5. Graph the solution set for each linear inequality.
   a) \( y > -2x + 8 \)
   b) \(-3y \leq 9x + 12 \)
   c) \( y < 6 \)
   d) \(-4x - 8 > 4 \)
   e) \(10x - 12 < -y \)
   f) \(4x + 3y \geq -12 \)
6. Graph the solution set for each linear inequality.
   a) \(\{(x, y) \mid 2x - y \geq 5y + 2x + 12, x \in \mathbb{W}, y \in \mathbb{W}\}\)
   b) \(\{(x, y) \mid x + 6y - 14 < 0, x \in \mathbb{I}, y \in \mathbb{I}\}\)
   c) \(\{(x, y) \mid 5x - y \leq 4, x \in \mathbb{W}, y \in \mathbb{W}\}\)
   d) \(\{(x, y) \mid 2x + 2 \leq 5 + x, x \in \mathbb{I}, y \in \mathbb{I}\}\)
   e) \(\{(x, y) \mid -2y > 20, x \in \mathbb{R}, y \in \mathbb{R}\}\)
   f) \(\{(x, y) \mid 4x - 5y < 10, x \in \mathbb{R}, y \in \mathbb{R}\}\)

7. Grace’s favourite activities are going to the movies and skating with friends. She budgets herself no more than $75 a month for entertainment and transportation. Movie admission is $9 per movie, and skating costs $5 each time. A student bus pass for the month costs $25.
   a) Define the variables and write a linear inequality to represent the situation.
   b) What are the restrictions on the variables? How do you know?
   c) Graph the linear inequality. Use your graph to determine:
      i) a combination of activities that Grace can afford and still have some money left over
      ii) a combination of activities that she can afford with no money left over
      iii) a combination of activities that will exceed her budget

8. Eamon coaches a hockey team of 18 players. He plans to buy new practice jerseys and hockey sticks for the team. The supplier sells practice jerseys for $50 each and hockey sticks for $85 each. Eamon can spend no more than $3000 in total. He wants to know how many jerseys and sticks he should buy.
   a) Write a linear inequality to represent the situation.
   b) Use your inequality to model the situation graphically.
   c) Determine a reasonable solution to meet the needs of the team, and provide your reasoning.

9. For every teddy bear that is sold at a fundraising banquet, $10 goes to charity. For every ticket that is sold, $32 goes to charity. The organizers’ goal is to raise at least $5000. The organizers need to know how many teddy bears and tickets must be sold to meet their goal.
   a) Define the variables and write a linear inequality to represent the situation.
   b) What are the restrictions on the variables? How do you know?
   c) Graph the linear inequality to help you determine whether each of the following points is in the solution set. The first coordinate is the number of teddy bears and the second is the number of tickets.
      i) (400, 20)  
      ii) (205, 98)  
      iii) (156, 105)
10. On Earth Day, a nursery sold more than $1500 worth of maple and birch trees. The maple trees were sold for $75, and the birch trees were sold for $50.
   a) Define the variables and write a linear inequality to represent the possible combinations of trees sold. Are there any restrictions on the variables? Explain.
   b) Graph the linear inequality.
   c) Use your graph to determine:
      i) if the nursery could have sold 13 of each type of tree
      ii) if 14 of one type and 9 of the other type could have been sold

11. In the fall, Javier plants tulip and crocus bulbs. Each tulip takes up an area of at least 12 square inches, and each crocus takes up an area of at least 9 square inches. Javier has a total area of 36 in. by 50 in., and he wants to plant at least 30 of each type of flower. He wants to know exactly how many of each type of flower he should plant.
   a) If you were to graph Javier’s situation, would the boundary be a dashed line, a stippled line, or a solid line? How do you know?
   b) Graph the linear inequality, and determine a reasonable solution to Javier’s problem.

12. A banquet room is set up to seat, at most, 660 people. Each rectangular table seats 12 people, and each circular table seats 8 people.
   a) Define the variables and write a linear inequality to represent the number of each type of table needed. Then graph your inequality.
   b) The organizers of the banquet would like to have as close to the same number of rectangular tables and circular tables as possible. What combination of tables could they use? Explain your choice.

Closing

13. Joelle used deductive reasoning to conclude that the graph on the right represents a linear inequality.
   a) What evidence has she used to arrive at this conclusion?
   b) State some other things you know about this inequality. Provide your reasoning for each.

Extending

14. Debbie and Gavin are moving. They have household goods that occupy a volume of, at most, 162 cubic feet. Packing boxes are available in two sizes: 4 cubic feet and 6 cubic feet. They are sold in sets of four.
   a) Use a graph to determine what combinations of boxes Debbie and Gavin could buy.
   b) If Debbie and Gavin wanted to use the least number of boxes, what is the best combination? Explain your thinking.
Chapter 6

6.2 Exploring Graphs of Systems of Linear Inequalities

YOU WILL NEED
- graphing technology OR graph paper, ruler, and coloured pencils

GOAL
Explore graphs of situations that can be modelled by systems of two linear inequalities in two variables.

EXPLORE the Math
A nursery school serves morning and afternoon snacks to its students. The morning snacks are fruits, vegetables, and juice, and the afternoon snacks are cheese and milk.
- The school can accommodate 50 students or fewer altogether. Students can attend for just the morning or for a full day.
- The morning snack costs $1 per student per week, and the afternoon snack costs $2 per student per week.
- The weekly snack budget is $120 or less.

What combinations of morning and full-day students can the school accommodate and stay within the weekly snack budget?

Reflecting
A. In a small group, compare the strategies that could be used to solve the problem. Would each strategy allow you to identify all the possible solutions to the problem? Explain.
B. How could a system of linear inequalities represent all the possible solutions?
C. Suggest a possible combination of morning and full-day students that would solve the problem. Explain your choice.
D. How would the graph of the system of inequalities change in each situation?
   i) The snack budget is $126 per week.
   ii) The school accommodates 30 or fewer students.
   iii) The school has 50 or more students, and the weekly snack budget is no more than $48.

system of linear inequalities
A set of two or more linear inequalities that are graphed on the same coordinate plane; the intersection of their solution regions represents the solution set for the system.
In Summary

Key Ideas

- Some contextual situations can be modelled by a system of two or more linear inequalities.
- All of the inequalities in a system of linear inequalities are graphed on the same coordinate plane. The region where their solution regions intersect or overlap represents the solution set to the system. For example, this graph shows the solution region to this system:

\[
\begin{align*}
&\{(x, y) \mid y \geq x, x \in \mathbb{R}, y \in \mathbb{R}\} \\
&\{(x, y) \mid y \leq 5, x \in \mathbb{R}, y \in \mathbb{R}\}
\end{align*}
\]

Need to Know

- As with the solution region for a single linear inequality, the solution region for a system of linear inequalities can be discrete or continuous and can be restricted to certain quadrants. For example, the graph to the right shows the system described below:

\[
\begin{align*}
&\{(x, y) \mid y \geq 1, x \in \mathbb{W}, y \in \mathbb{W}\} \\
&\{(x, y) \mid y \leq -3x + 6, x \in \mathbb{W}, y \in \mathbb{W}\}
\end{align*}
\]

Its solution region is restricted to discrete points with whole-number coordinates in the first quadrant.
- If the solution regions for the linear inequalities in the system do not overlap, there is no solution.

FURTHER Your Understanding

1. Three systems of linear inequalities have been graphed below. For each system, describe what you can infer from the graph about the restrictions on the domain and range.

   a) \(y \geq -2x\)  
      \(-3 < x\)

   b) \(x + 3y \geq 0\)  
      \(x + y \geq 2\)

   c) \(x + y \leq -2\)  
      \(2y \geq x\)

2. Graph each system of linear inequalities. Justify your representation of the solution set.

   a) \(\{(x, y) \mid -x + 2y \geq -4, x \in \mathbb{R}, y \in \mathbb{R}\}\)
      \(\{(x, y) \mid y \geq x, x \in \mathbb{R}, y \in \mathbb{R}\}\)

   b) \(\{(x, y) \mid 2x + 3y \leq 9, x \in \mathbb{I}, y \in \mathbb{I}\}\)
      \(\{(x, y) \mid y - 6x \geq 1, x \in \mathbb{I}, y \in \mathbb{I}\}\)
6.3 Graphing to Solve Systems of Linear Inequalities

**YOU WILL NEED**
- graphing technology OR graph paper, ruler, and coloured pencils

**GOAL**
Solve problems by modelling systems of linear inequalities.

**LEARN ABOUT the Math**
A company makes two types of boats on different assembly lines: aluminum fishing boats and fibreglass bow riders.
- When both assembly lines are running at full capacity, a maximum of 20 boats can be made in a day.
- The demand for fibreglass boats is greater than the demand for aluminum boats, so the company makes at least 5 more fibreglass boats than aluminum boats each day.

? What combinations of boats should the company make each day?

**EXAMPLE 1**
Solving a problem with discrete whole-number variables using a system of inequalities.

Mary’s Solution: Using graph paper

Let \(a\) represent the number of aluminum fishing boats. Let \(f\) represent the number of fibreglass bow riders.

\[
\begin{align*}
a & \in \mathbb{W} \text{ and } f \in \mathbb{W} \\
\end{align*}
\]

The relationship between the two types of boats can be represented by this system of inequalities:

\[
\begin{align*}
a + f & \leq 20 \\
a + 5 & \leq f
\end{align*}
\]

I knew I could solve this problem by representing the situation algebraically with a system of two linear inequalities and graphing the system.

Since only complete boats are sold, I knew that \(a\) and \(f\) are whole numbers and the graph would consist of discrete points in the first quadrant.

The two inequalities describe:
- a combination of boats to a maximum of 20.
- at least 5 more fibreglass boats than aluminum boats.
To graph each linear inequality, I knew I had to graph its boundary as a stippled line, and then shade and stipple the correct half plane.

To graph each boundary, I wrote each linear equation and then determined the \( a \)- and \( f \)-intercepts so I could plot and join them.

For \( a + 5 = f \), I knew \((-5, 0)\) wasn’t going to be a point on the boundary, because it’s not in the first quadrant, so I chose another point by solving the equation for \( a = 5 \).

I tested point \((0, 0)\) to determine which half plane to shade for \( a + 5 \leq f \).

I drew a green stippled boundary connecting \((0, 20)\) and \((20, 0)\) and shaded the half plane below it orange, because the solution region is discrete.

I tested \((0, 0)\) to determine which half plane to shade for \( a + f \leq 20 \).

To graph each linear inequality, I knew I had to graph its boundary as a stippled line, and then shade and stipple the correct half plane.

To graph each boundary, I wrote each linear equation and then determined the \( a \)- and \( f \)-intercepts so I could plot and join them.

For \( a + 5 = f \), I knew \((-5, 0)\) wasn’t going to be a point on the boundary, because it’s not in the first quadrant, so I chose another point by solving the equation for \( a = 5 \).

I tested point \((0, 0)\) to determine which half plane to shade for \( a + 5 \leq f \).

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<thead>
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<th>( a + f )</th>
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<tbody>
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<td>20</td>
<td>( a + f )</td>
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<td>0 + 0</td>
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Since \( 0 \leq 20 \), \((0, 0)\) is in the solution region.

| \( a + f \leq 20 \) |
|---|---|
| \((0, 20)\) | \((20, 0)\) |

Test \((0, 0)\) in \( a + f \leq 20 \).

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\( a + 5 \leq f \)

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<tr>
<td>0 + 5</td>
<td>0</td>
</tr>
</tbody>
</table>

Since \( 5 \) is not less than or equal to \( 0 \), \((0, 0)\) is not in the solution region.
I plotted the points (0, 5) and (5, 10) on the same coordinate plane. I used these points to draw a green stippled boundary for \( a + 5 \leq f \).

I shaded the half plane above the boundary orange, since the test point (0, 0) is not a solution to the linear inequality and the solution region is discrete.

I knew that the solution set for the system of linear inequalities is represented by the intersection or overlap of the solution regions of the two inequalities. This made sense since points in this region satisfy both inequalities.

I knew that the triangular solution region included discrete points along its three boundaries, including the \( y \)-axis from \( y = 5 \) to \( y = 10 \).

Since the solution set for the system contains only discrete points with whole-number coordinates, I stippled its solution region.

I knew that any whole-number point in the triangular solution region is a possible solution. For example, (3, 12) is a possible solution.

\[
\{(a, f) \mid a + f \leq 20, a \in \mathbb{W}, f \in \mathbb{W}\}
\]
\[
\{(a, f) \mid a + 5 \leq f, a \in \mathbb{W}, f \in \mathbb{W}\}
\]

Any point with whole-number coordinates in the intersecting or overlapping region is an acceptable combination. For example, 3 aluminum boats and 12 fibreglass boats is an acceptable combination.

I knew that (3, 12) worked because this gives a total of 15 boats with 9 more fibreglass boats than aluminum boats.
Liv’s Solution: Using graphing technology

Let \( x \) represent the number of aluminum fishing boats. Let \( y \) represent the number of fibreglass bow riders.

\[ x \in \mathbb{W} \text{ and } y \in \mathbb{W} \]

The relationship between the two types of boats can be represented by this system of inequalities:

\[
\begin{align*}
    x + y &\leq 20 \\
    x + 5 &\leq y \\
    y &\leq 20 - x \\
    y &\leq -x + 20
\end{align*}
\]

Any point with whole-number coordinates in the intersecting or overlapping solution region is an acceptable combination. For example, 5 aluminum boats and 15 fibreglass boats is an acceptable combination:

\[
\begin{align*}
    x + y &\leq 20 \\
    (5) + (15) &\leq 20 \\
    20 &\leq 20
\end{align*}
\]

This is valid.

Reflecting

A. Is every point on the boundaries of the solution region a possible solution? Explain.

B. Are the three points where the boundaries intersect part of the solution region? Explain.

C. How would the graph change if fewer than 25 boats were made each day?

D. All points with whole-number coordinates in the solution region are valid, but are they all reasonable? Explain.
**APPLY the Math**

**EXAMPLE 2** Solving graphically a system of two linear inequalities with continuous variables

Graph the solution set for the following system of inequalities. Choose two possible solutions from the set.

\[3x + 2y > -6\]
\[y \leq 3\]

**Peter’s Solution: Using graph paper**

\[x \in \mathbb{R}, y \in \mathbb{R}\]

\[3x + 2y > -6\]

x-intercept: \[3x + 2(0) = -6\]
\[3x = -6\]
\[x = -2\]
\[(-2, 0)\]

y-intercept: \[3(0) + 2y = -6\]
\[2y = -6\]
\[y = -3\]
\[(0, -3)\]

Test \((0, 0)\) in \(3x + 2y > -6\).

<table>
<thead>
<tr>
<th>LS</th>
<th>RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3x + 2y)</td>
<td>(-6)</td>
</tr>
<tr>
<td>(3(0) + 2(0))</td>
<td>0</td>
</tr>
</tbody>
</table>

Since \(0 > -6\), \((0, 0)\) is in the solution region.

I assumed both \(x\) and \(y\) are in the set of real numbers because restrictions on the domain and range were not stated. I knew the graph would have a continuous solution region and could be in all four quadrants.

To graph \(3x + 2y > -6\), I identified the \(x\)- and \(y\)-intercepts of the linear equation of the boundary \(3x + 2y = -6\).

I used the test point \((0, 0)\) to determine which region to shade.
Any point in the solution region is a possible solution. The overlapping solution region represents the solution set of the system of linear inequalities. Therefore, (2, -3) and (-1.5, 3) are two possible solutions.
Rita’s Solution: Using graphing technology

\[ x \in \mathbb{R}, y \in \mathbb{R} \]
\[ y \leq 3 \]
\[ 3x + 2y > -6 \]
\[ \frac{2y}{2} > -\frac{3x}{2} - \frac{6}{2} \]
\[ y > -\frac{3x}{2} - 3 \]

To enter the inequalities into my graphing calculator, I isolated \( y \).

I assumed both \( x \) and \( y \) are in the set of real numbers because no restrictions were stated. I knew the graph would have a continuous solution region and could be in all four quadrants.

I set my calculator window to show all four quadrants, since \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \).

I entered \( y > -\frac{3x}{2} - 3 \) and \( y \leq 3 \), and then graphed the system.

I interpreted the overlapping region as follows:
- The solution region is where the solution regions of the two inequalities intersect.
- Only solid boundaries are included in the solution region.
- The point where the dashed line intersects the solid line is not part of the solution.
- All the points in the solution region represent the solution set for the system because \( x \) and \( y \) are real numbers.

The overlapping solution region represents the solution set of the system of inequalities.

Therefore, \((2, 2)\) and \((-0.5, -1.7)\) are two possible solutions.

The solution region is a possible solution.

Any point in the solution region is a possible solution.

Your Turn

How would the solution region change if \( x \in \mathbb{I} \) and \( y \in \mathbb{I} \)? How would it stay the same?
6.3 Graphing to Solve Systems of Linear Inequalities

EXAMPLE 3 | Solving graphically a problem with continuous positive variables

A sloop is a sailboat with two sails: a mainsail and a jib. When a sail is fully out or up, it is said to be “out 100%.” When the winds are high, sailors often reef, or pull in, the sails to be less than their full capability.

- Jim is sailing in winds of 22 knots, so he wants no more than 80% of the mainsail out.
- Jim also wants more mainsail out than jib.

What possible combinations of mainsail and jib can Jim have out?

Louise’s Solution: Using graph paper

Let \( m \) represent the percent of mainsail out.
Let \( j \) represent the percent of jib out.

\[ m \geq 0 \text{ and } j \geq 0, \text{ where } m \in \mathbb{R}, j \in \mathbb{R} \]

The relationship between the two types of sails can be represented by the following system of two linear inequalities:

\[ m \leq 80 \]
\[ j < m \]

\[ m \leq 80 \]
Boundary: \( m = 80 \)
Boundary is a vertical line with an \( m \)-intercept of 80.

\[ j < m \]
Boundary: \( j = m \)
Boundary line has a slope of 1 and a \( j \)-intercept of 0.

The inequalities describe the following information:
- No more than 80% of the mainsail can be out.
- Less jib than mainsail must be out.

I decided to use \( m \) as the independent variable.
I examined each inequality to determine its boundary:
- Since \( m \) is the independent variable, I knew the boundary for \( m \leq 80 \) would be a vertical line through \( m = 80 \).
- I knew the boundary of \( j < m \) has a slope of 1 and passes through the point \((0, 0)\).
For \( m \leq 80 \), I drew a solid green vertical line through \( m = 80 \) and then shaded the half plane to its left green, since the inequality sign is \( \leq \).

For \( j < m \), I drew a green dashed line through \((0, 0)\) with a slope of 1 and I shaded the half plane below green since the inequality is <.

I drew an open dot where the dashed boundary intersects the solid boundary to show that point isn’t part of the solution region.

The solution region for the system is a right triangle and consists of all the points in the overlapping region, including the solid boundary and the \( m \)-axis from 0 to 80.

\[
\{(m, j) \mid m \leq 80, m \geq 0, j \geq 0, m \in \mathbb{R}, j \in \mathbb{R}\}
\]

\[
\{(m, j) \mid j < m, m \geq 0, j \geq 0, m \in \mathbb{R}, j \in \mathbb{R}\}
\]

Any point in the solution region represents an acceptable combination. For example,

- 80% of the mainsail and 70% of the jib can be out.
- 70% of the mainsail and 40% of the jib can be out.
- 40.5% of the mainsail and 30.75% of the jib can be out.

**Your Turn**

Suppose Jim wanted to have

- at least twice as much jib as mainsail out, and
- no more than 90% of the jib out.

How would the solution region change?
In Summary

Key Ideas
- When graphing a system of linear inequalities, the boundaries of its solution region may or may not be included, depending on the types of linear inequalities (≥, ≤, <, or >) in the system.
- Most systems of linear inequalities representing real-world situations are restricted to the first quadrant because the values of the variables in the system must be positive.

Need to Know
- Any point in the solution region for a system is a valid solution, but some solutions may make more sense than others depending on the context of the problem.
- You can validate a possible solution from the solution region by checking to see if it satisfies each linear inequality in the system. For example, to validate if (2, 2) is a solution to the system:
  \[ x + y \geq 1 \\
  2 > x - 2y \]
  Validating (2, 2) for \[ x + y \geq 1 \]:
  \[
  \begin{array}{c|c}
  \text{LS} & \text{RS} \\
  \hline
  x + y & 1 \\
  2 + 2 & 4 \\
  4 \geq 1 & \text{valid}
  \end{array}
  \]
  Validating (2, 2) for \[ 2 > x - 2y \]:
  \[
  \begin{array}{c|c}
  \text{LS} & \text{RS} \\
  \hline
  2 & x - 2y \\
  2 & 2 \\
  2 > 2 & \text{valid}
  \end{array}
  \]
- Use an open dot to show that an intersection point of a system’s boundaries is excluded from the solution set. This occurs when a dashed line intersects a dashed or solid line.
- Use a solid dot to show that an intersection point of a system’s boundaries is included in the solution set. This occurs when both boundary lines are solid.

CHECK Your Understanding

1. Graph the solution set for each system of inequalities. Determine a solution. Check its validity.
   a) \[ x + y \geq 1 \]
   b) \[ x + 2y < 6 \]
   c) \[ 2x - 4 \geq y \]
   i) \[ x \leq 4 \]
   ii) \[ x < 4 \]
   iii) \[ x < y \]
   iv) \[ 2y + 3x \leq 7 \]

2. a) Graph the solution set for this system of inequalities. Determine a solution. Check its validity. Describe the solution region.
   \[ x \leq 6 \]
   \[ 3y - x < 6 \]
   b) Determine if each point is in the solution region.
   i) \( (6, 4) \)  ii) \( (8, 2) \)  iii) \( (3, 2) \)  iv) \( (3, 3) \)
3. For each system of linear inequalities, explain whether the boundaries and their points of intersection are part of the solution region.
   a) \( \{(x, y) \mid y \leq -2x, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   \( \{(x, y) \mid 3 \leq -x - y, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   b) \( \{(x, y) \mid x + y \leq -2, x \in \mathbb{I}, y \in \mathbb{I}\} \)
   \( \{(x, y) \mid 2y \geq x, x \in \mathbb{I}, y \in \mathbb{I}\} \)
   c) \( \{(x, y) \mid x + 3y \geq 0, x \in \mathbb{I}, y \in \mathbb{I}\} \)
   \( \{(x, y) \mid x + y > 2, x \in \mathbb{I}, y \in \mathbb{I}\} \)

PRACTISING

4. Graph each system. Determine a solution for each.
   a) \( \{(x, y) \mid x + y \leq 3, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   \( \{(x, y) \mid y > 2, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   b) \( \{(x, y) \mid 2x + y > 0, x \in \mathbb{W}, y \in \mathbb{W}\} \)
   \( \{(x, y) \mid y > x, x \in \mathbb{W}, y \in \mathbb{W}\} \)
   c) \( \{(x, y) \mid 3y - 2x \leq 6, x \in \mathbb{N}, y \in \mathbb{N}\} \)
   \( \{(x, y) \mid 2y - 3x \leq 6, x \in \mathbb{I}, y \in \mathbb{I}\} \)
   d) \( \{(x, y) \mid y - x \geq 2, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   \( \{(x, y) \mid y + 2 \leq x, x \in \mathbb{R}, y \in \mathbb{R}\} \)

5. For each system of linear inequalities,
   a) graph the solution set.
   b) describe the solution region.
   c) determine a solution. Check its validity.
   i) \( \{(x, y) \mid 3x + y > 5, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   \( \{(x, y) \mid y \leq 4, x \in \mathbb{R}, y \in \mathbb{R}\} \)
   ii) \( \{(x, y) \mid x + y \geq 1, x \in \mathbb{N}, y \in \mathbb{N}\} \)
   \( \{(x, y) \mid 2 > x - 2y, x \in \mathbb{N}, y \in \mathbb{N}\} \)
   iii) \( \{(x, y) \mid 4x + 11y > 44, x \in \mathbb{W}, y \in \mathbb{W}\} \)
   \( \{(x, y) \mid 2x - 6 \leq 3y, x \in \mathbb{W}, y \in \mathbb{W}\} \)

6. The staff in a cafeteria are making two kinds of sandwiches: egg salad, and ham and cheese:
   - A maximum of 450 sandwiches are needed.
   - Based on previous demand, there should be at least twice as many ham and cheese sandwiches as egg salad sandwiches.
   a) Define the variables and write a system of inequalities that models this situation.
   b) Describe the restrictions on the variables in this situation.
   c) Graph the system to determine the solution set.
   d) Suggest two combinations of numbers of sandwiches that the cafeteria staff could make.
7. **a)** Graph the solution set for the following system of inequalities. Determine a solution. Check its validity.

\[
9x + 18y < 18 \\
3x - 6y \leq 18
\]

**b)** Is each point below a possible solution to the system? How do you know?

i) (4, -1)  
ii) (-2, 2)  
iii) (-4, -2)  
iv) (9, 1)  
v) (-2.5, -1.5)  
vi) (2, -2)

8. Trish is setting up her social networking page:

- She wants to have no more than 500 friends on her new social networking page.
- She also wants to have at least three school friends for every rugby friend.

**a)** Define the variables and write a system of inequalities that models this situation.

**b)** Describe the restrictions on the domain and range of the variables.

**c)** Graph the solution set to determine two possible combinations of school friends and rugby friends she could have.

9. Graph the solution set for this system of linear inequalities to determine two valid solutions:

\[
\{(x, y) \mid 3x + y \leq 2, x \in \mathbb{I}, y \in \mathbb{I}\} \\
\{(x, y) \mid 2y + 3x > 1, x \in \mathbb{I}, y \in \mathbb{I}\}
\]

10. Spence, a disc jockey, is often hired to play weddings.

- His contract states that he will play no longer than 3 h, with no more than 12 songs each hour.
- He likes to play two or more songs for young listeners for every one song he plays for older listeners.

Determine three possible combinations of numbers of songs he could play.

**Closing**

11. Create a system of two linear inequalities for each situation.

**a)** The boundaries of the solution region are included in the solution set, and all the points in the solution region are valid solutions.

**b)** The boundaries of the solution region are not included in the solution set, and only whole-number points in the solution region are valid solutions.

**Extending**

12. Graph the solution set for the following system of inequalities.

Describe the solution region.

\[
2y + 8 \leq x \\
-4 + y + x < 0 \\
3x + y + 3 > 3(1 + x)
\]
Applying Problem-Solving Strategies

Inequality Sudoku

Sudoku puzzles are seen in many books, newspapers, and magazines. However, inequality Sudoku puzzles, like the one shown below, are much less common.

The Puzzle

A. Complete the puzzle by inserting the digits 1 through 6 into the cells so that the inequality signs are correct. Each digit must occur only once in each column or row, as in a conventional Sudoku puzzle. Also, the digits 1 to 6 must occur within each region outlined in red.

The Strategy

B. Describe what you know about the rows and columns and the given numbers.

C. Describe how you used deductive reasoning to complete the puzzle.

Variation

D. Start with a blank six-by-six grid. Fill in the digits 1 to 6 so that each digit occurs only once in each row and column. Place inequality signs between the digits. This completed puzzle is your answer key.

E. Prepare a blank copy of your six-by-six grid, with just the inequality signs and a few numbers.

F. Switch puzzles with a classmate, and solve each other’s puzzles using deductive reasoning.

G. If you are unable to find a starting point, ask your classmate to place a 6 or a 1 in its correct place in the puzzle.
FREQUENTLY ASKED Questions

Q: When graphing a linear inequality, what process should you use?

A: When graphing on graph paper, use the following process:

**Step 1.** Decide how much of the coordinate plane to include by determining the domain and range of the inequality.

**Step 2.** Graph the boundary:
- Determine two points that are solutions to the equation of the boundary.
- Plot the points to draw the boundary:
  - Make the boundary a solid green line if equality is possible (\( \geq \) and \( \leq \)) and the solution set is continuous.
  - Draw a dashed line if equality is not possible (\(<\) or \(>\)). Use green if the solution set is continuous and orange if it is discrete.
  - If equality is possible (\( \leq \) or \( \geq \)) and the solution set is discrete, plot green points to stipple the boundary.

**Step 3.** Shade the appropriate half plane:
- You can test a point on either side of the boundary to see if it is in the solution region for the linear inequality. Point \((0, 0)\) is often used.
  - If the solution set is continuous, shade the correct half plane green.
  - If the solution set is discrete, shade the correct half plane orange and then plot green points that have integer, whole number, or natural-number coordinates (as appropriate) to stipple it.

When using graphing technology to graph, use this process:

**Step 1.** Decide how much of the coordinate plane to include by determining the domain and range.

**Step 2.** Isolate \(y\) in the linear inequality.

**Step 3.** Graph the linear inequality.

**Step 4.** Verify that the linear inequality has been represented correctly:
- Use a test point, such as \((0, 0)\), to check that the correct half plane is shaded.
- Check the boundary against the inequality sign.
Q: How can you determine if all the points or only some of the points in the solution region are part of the solution set?
A: Whether a solution set for a system of linear inequalities includes discrete or continuous values depends on the restrictions placed on the domain and range of the variables. Consider these examples:

Graph the solution set for this system of linear inequalities:
\[ y \leq -x + 10 \]
\[ y \geq x - 6 \]

Starr is mixing green, \( g \), and dark purple dye, \( d \), to make a brown dye. She needs less than 100 mL of the brown dye. She wants to use at least 20 mL more of the green dye than the dark purple dye. What combinations of dye are possible?
\[ g + d < 100 \]
\[ g \geq d + 20 \]

A group has fewer than 10 students, and there are at least 2 more girls, \( g \), than boys, \( b \). What combinations of girls and boys are possible?
\[ g + b < 10 \]
\[ g \geq b + 2 \]

Since no context is given and the domain and range are not stated, the variables are assumed to be real numbers and the graph could be in all four quadrants. The solution set is represented by all points in the solution region, which includes the solid boundaries.

Measurements are continuous. The solution set is represented by all points in the solution region in the first quadrant, which includes the solid boundary, but not the intersection of the two boundaries.

Numbers of students are whole numbers. The solution set is represented by points with whole-number coordinates in the solution region, which includes the stippled boundary, but not the intersection of the two boundaries.
1. For a school fundraiser, the drama students are selling white and dark chocolates. The goal is to sell at least 70 kg of chocolates, in total, and they need to determine how many kilograms of each to buy.
   a) Define the variables and write an inequality to model this situation.
   b) Graph the inequality. Use the graph to choose three possible combinations of kilograms of white and dark chocolates. Explain your choices.

2. a) What can you deduce from this graph of a linear inequality?
   b) Is each point a possible solution?
      i) (2, 2)
      ii) (3.5, -1)
      iii) (-2, -2)
      iv) (−2, 2)

3. Horst and Lev volunteer at a seniors’ centre. Together they volunteer, at most, 30 h each week and work only a whole number of hours.
   a) Define the variables and write an inequality to model this situation.
   b) Graph the inequality and use it to find several possible combinations of hours that the two boys could volunteer.

Lesson 6.2

4. Nick is preparing a tomato and red pepper soup as the daily special for his restaurant.
   • To allow the red pepper taste to dominate, he will include at least twice as many peppers as tomatoes, by mass.
   • However, he wants no more than 25 kg of tomatoes and red peppers altogether.
   a) Define the variables and write a system of inequalities to model this situation.
   b) Graph the system. Use your graph to suggest three possible combinations of tomatoes and peppers.

Lesson 6.3

5. a) Graph this system of linear inequalities:
   $\{(x, y) \mid x \leq 3y, x \in \mathbb{W}, y \in \mathbb{W}\}
   \{(x, y) \mid x + y \leq 60, x \in \mathbb{W}, y \in \mathbb{W}\}$
   b) Suggest a possible context for this system, and explain why you chose this context.

6. A flag is being created for a soccer team.
   • The length must be less than 100 cm.
   • The perimeter must be 400 cm or less.
   Use a graph to choose three possible combinations of length and width. Explain your choices.

Lesson 6.2

7. A service station owner, Uma, has two part-time employees: Pali and Meg.
   • Pali is skilled at repairs but has limited experience with customers. Uma pays him $18 an hour.
   • Meg has experience with customers but can do only simple repairs. Uma pays her $10 an hour.
   • Uma has a budget of $470 for their wages.
   • Uma can hire both of these employees for no more than 30 h a week, in total. Both employees are scheduled in whole numbers of hours.
   a) Use a graph to choose two possible combinations of hours for Pali and Meg. Explain your choices.
   b) For each change in the situation below, predict how the graph would change. Explain your prediction, and then graph to check it.
      i) Uma’s budget is $400 a week.
      ii) Uma wants Pali to work at least twice as many hours as Meg.
Optimization Problems I: Creating the Model

INVESTIGATE the Math

A toy company manufactures two types of toy vehicles: racing cars and sport-utility vehicles.

- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs $8 to make a racing car and $12 to make a sport-utility vehicle. There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.

How can this situation be modelled?

A. What are the two variables in this situation?
B. Write a system of three linear inequalities to represent these conditions:
   - the total number of racing cars that can be made
   - the total number of sport-utility vehicles that can be made
   - the total number of vehicles that can be made

<table>
<thead>
<tr>
<th>Baby's Breath</th>
<th>Bracken Fern</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>114.5</td>
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<tr>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>150</td>
<td>125</td>
</tr>
</tbody>
</table>
C. What do you know about the restrictions on the domain and range of the variables? Explain.

D. Graph the system. Choose at least two points in the solution region that are possible solutions to the system.

E. What quantity in this situation needs to be minimized and maximized? Write an equation to represent how the two variables relate to this quantity.

Reflecting

F. Each combination below is a possible solution to the system of linear inequalities:
   i) 40 racing cars and 60 sport-utility vehicles
   ii) 40 racing cars and 30 sport-utility vehicles
   iii) 10 racing cars and 60 sport-utility vehicles
   iv) 30 racing cars and 40 sport-utility vehicles

Use your equation from part E to calculate the manufacturing cost for each solution. What do you notice?

**APPLY the Math**

**EXAMPLE 1** Creating a model for an optimization problem with whole-number variables

Three teams are travelling to a basketball tournament in cars and minivans.
- Each team has no more than 2 coaches and 14 athletes.
- Each car can take 4 team members, and each minivan can take 6 team members.
- No more than 4 minivans and 12 cars are available.

The school wants to know the combination of cars and minivans that will require the minimum and maximum number of vehicles. Create a model to represent this situation.

**Juanita’s Solution**

Let \( m \) represent the number of minivans.
Let \( c \) represent the number of cars.

\[
m \in \mathbb{W} \quad \text{and} \quad c \in \mathbb{W}
\]
**optimization problem**
A problem where a quantity must be maximized or minimized following a set of guidelines or conditions.

**constraint**
A limiting condition of the optimization problem being modelled, represented by a linear inequality.

**objective function**
In an optimization problem, the equation that represents the relationship between the two variables in the system of linear inequalities and the quantity to be optimized.

**feasible region**
The solution region for a system of linear inequalities that is modelling an optimization problem.

---

I created an equation, called the **objective function**, to represent the relationship between the two variables (number of minivans and number of cars) and the quantity to be minimized and maximized.

Objective function:
Let \( V \) represent the total number of vehicles.
\[ V = c + m \]

Constraints:
Number of cars available:
\[ c \leq 12 \]
Number of minivans available:
\[ m \leq 4 \]
Number of team members:
\[ 4c + 6m \leq 48 \]

I knew that this is an **optimization problem** because the number of vehicles has to be minimized and maximized.

I wrote three linear inequalities to represent the three limiting conditions, or **constraints**.

The maximum number of team members is the number of teams multiplied by the maximum number of coaches and athletes:
\[ 3(14) + 3(2) = 48 \]

I created an equation, called the **objective function**, to represent the relationship between the two variables (number of minivans and number of cars) and the quantity to be minimized and maximized (number of vehicles).

I graphed the system of three inequalities.

One of the solutions in the **feasible region** represents the combination of cars and minivans that results in the minimum total number of vehicles and another solution represents the maximum. I think I could use the objective function to determine each point, but I am not certain how yet.

---

**Your Turn**
Suppose that the greatest number of athletes changed from 14 to 12 per team. How would Juanita’s model change?
**EXAMPLE 2**  Creating a model for a maximization problem with positive real-number variables

A refinery produces oil and gas.
- At least 2 L of gasoline is produced for each litre of heating oil.
- The refinery can produce up to 9 million litres of heating oil and 6 million litres of gasoline each day.
- Gasoline is projected to sell for $1.10 per litre.
  Heating oil is projected to sell for $1.75 per litre.

The company needs to determine the daily combination of gas and heating oil that must be produced to maximize revenue. Create a model to represent this situation.

**Umberto’s Solution**

Let \( h \) represent the number of litres of heating oil.
Let \( g \) represent the number of litres of gasoline.

Restrictions:
\[ h \geq 0 \text{ and } g \geq 0, \text{ where } h \in \mathbb{R} \text{ and } g \in \mathbb{R}. \]

Constraints:
- Ratio of gasoline produced to oil produced:
  \[ g \geq 2h \]
- Amount of gasoline that can be produced:
  \[ g \leq 6,000,000 \]
- Amount of oil that can be produced:
  \[ h \leq 9,000,000 \]
Let $R$ represent total revenue from sales of gasoline and heating oil.

Objective function to maximize:

$$R = 1.10g + 1.75h$$

Your Turn

Suppose that the refinery produced at least 2 L of heating oil for each litre of gasoline. Discuss with a partner how Umberto’s model would change.
In Summary

Key Ideas

- To solve an optimization problem, you need to determine which combination of values of two variables results in a maximum or minimum value of a related quantity.
- When creating a model, the first step is to represent the situation algebraically. An algebraic model includes these parts:
  - a defining statement of the variables used in your model
  - a statement describing the restrictions on the variables
  - a system of linear inequalities that describes the constraints
  - an objective function that shows how the variables are related to the quantity to be optimized
- The second step is to represent the system of linear inequalities graphically.
- In optimization problems, any restrictions on the variables are considered constraints. For example, if you are working with positive real numbers, $x \geq 0$ and $y \geq 0$ are constraints and should be included in the system of linear inequalities.

Need to Know

- You can create a model for an optimization problem by following these steps:
  - **Step 1.** Identify the quantity that must be optimized. Look for key words, such as maximize or minimize, largest or smallest, and greatest or least.
  - **Step 2.** Define the variables that affect the quantity to be optimized. Identify any restrictions on these variables.
  - **Step 3.** Write a system of linear inequalities to describe all the constraints of the problem. Graph the system.
  - **Step 4.** Write an objective function to represent the relationship between the variables and the quantity to be optimized.
CHECK Your Understanding

1. Baskets of fruit are being prepared to sell.
   • Each basket contains at least 5 apples and at least 6 oranges.
   • Apples cost 20¢ each, and oranges cost 35¢ each. The budget allows no more than $7, in total, for the fruit in each basket.
   Answer each part below to create a model that could be used to determine the combination of apples and oranges that will result in the maximum number of pieces of fruit in a basket.
   a) What are the two variables in this situation? Describe any restrictions.
   b) Write a system of linear inequalities to represent each constraint:
      i) the number of apples in each basket
      ii) the number of oranges in each basket
      iii) the cost of each basket (in cents)
   c) Graph the system.
   d) Write the objective function that represents how the quantity to be maximized relates to the variables.

2. A fast-food concession stand sells hotdogs and hamburgers.
   • Daily sales can be as high as 300 hamburgers and hot dogs combined.
   • The stand has room to stock no more than 200 hot dogs and no more than 150 hamburgers.
   • Hot dogs are sold for $3.25, and hamburgers are sold for $4.75.
   Create a model that could be used to determine the combination of hamburgers and hot dogs that will result in maximum sales.

PRACTISING

3. A vending machine sells juice and pop.
   • The machine holds, at most, 240 cans of drinks.
   • Sales from the vending machine show that at least 2 cans of juice are sold for each can of pop.
   • Each can of juice sells for $1.00, and each can of pop sells for $1.25.
   Create a model that could be used to determine the maximum revenue from the vending machine.

4. A student council is ordering signs for the spring dance. Signs can be made in letter size or poster size.
   • No more than 15 of each size are wanted.
   • At least 15 signs are needed altogether.
   • Letter-size signs cost $9.80 each, and poster-size signs cost $15.75 each.
   Create a model that could be used to determine a combination of the two sizes of signs that would result in the lowest cost to the council.
5. A football stadium has 50 000 seats.
   • Two-fifths of the seats are in the lower deck.
   • Three-fifths of the seats are in the upper deck.
   • At least 30 000 tickets are sold per game.
   • A lower deck ticket costs $120, and an upper deck ticket costs $80.
Create a model that could be used to determine a combination of
tickets for lower-deck and upper-deck seats that should be sold to
maximize revenue.

6. Sung and Faith have weekend jobs at a marina, applying anti-fouling
   paint to the bottom of boats.
   • Sung can work no more than 14 h per weekend.
   • Faith is available no more than 18 h per weekend.
   • The marina will hire both of them for 24 h or less per weekend.
   • Sung paints one boat in 3 h, but Faith needs 4 h to paint one boat.
   The marina wants to maximize the number of boats that are painted
each weekend.
   a) Create a model to represent this situation.
   b) Suppose that another employee, Frank, who can paint a boat in
   2 h, replaced Faith for a weekend. How would your model change?

7. A Saskatchewan farmer is planting wheat and barley.
   • He wants to plant no more than 1000 ha altogether.
   • The farmer wants at least three times as many hectares of wheat as barley.
   • The yield per hectare of wheat averages 50 bushels, and the yield per
   hectare of barley averages 38 bushels.
   • Wheat pays the farmer $5.25 per bushel, and barley pays $3.61 per
   bushel.
The farmer wants to plant a combination of wheat and barley that will
maximize revenue. Create a model to represent this situation.

Closing

8. With a partner, develop a list of questions that could help you create a
   model for an optimization problem.

Extending

9. At an orienteering meet, two courses are run over two days: a long
   course on the first day and a shorter course on the second day. George
   wants to run the first course in no more than 16 min and the second
course in no more than 12 min. He is aiming for a combined time of
no more than 25 min. The winner has the lowest combined time and
wins $25 for every minute under 30 min. Create a model to represent
this situation.
6.5

Optimization Problems II: Exploring Solutions

**YOU WILL NEED**
- graphing technology OR graph paper, ruler, and coloured pencils

**GOAL**
Explore the feasible region of a system of linear inequalities.

**EXPLORE the Math**

A toy company manufactures two types of toy vehicles: racing cars and sport-utility vehicles.
- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs $8 to make a racing car and $12 to make a sport-utility vehicle.

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.

The following model represents this situation. The feasible region of the graph represents all the possible combinations of racing cars \( r \) and sport-utility vehicles \( s \).

**Variables:**
- Let \( s \) represent the number of sport-utility vehicles.
- Let \( r \) represent the number of racing cars.
- Let \( C \) represent the cost of production.

**Restrictions:**
- \( s \in W, r \in W \)

**Constraints:**
- \( s \geq 0 \)
- \( r \geq 0 \)
- \( r \leq 40 \)
- \( s \leq 60 \)
- \( r + s \geq 70 \)

**Objective function to optimize:**
- \( C = 12s + 8r \)

How can you use patterns in the feasible region to predict the combinations of sport-utility vehicles and racing cars that will result in the minimum and maximum values of the objective function?
Reflecting

A. With a partner, discuss the pattern in the value of $C$ throughout the feasible region. Is the pattern what you expected? Explain.

B. As you move from left to right across the feasible region, what happens to the value of $C$?

C. As you move from the bottom to the top of the feasible region, what happens to the value of $C$?

D. What points in the feasible region result in each optimal solution? i) the maximum possible value of $C$ ii) the minimum possible value of $C$

E. Explain how you could verify that your solutions from part D satisfy each constraint in the model.

In Summary

Key Ideas

- The value of the objective function for a system of linear inequalities varies throughout the feasible region, but in a predictable way.
- The optimal solutions to the objective function are represented by points at the intersections of the boundaries of the feasible region. If one or more of the intersecting boundaries is not part of the solution set, the optimal solution will be nearby.

Need to Know

- You can verify each optimal solution to make sure it satisfies each constraint by substituting the values of its coordinates into each linear inequality in the system.
- The intersection points of the boundaries are called the vertices, or corners, of the feasible region.
1. Where might you find the maximum and minimum solutions to each objective function below? Explain how you know.

**a) Model A**
Restrictions:
\[ x \in \mathbb{R}, \ y \in \mathbb{R} \]
Constraints:
\[ x > -4 \]
\[ x - y \leq 8 \]
\[ y \leq 3 \]
Objective function:
\[ T = 2x + 5y \]

2. Consider the model below. What point in the feasible region would result in the minimum value of the objective function? How could you have predicted this from examining the objective function?

Restrictions:
\[ x \in \mathbb{R}, \ y \in \mathbb{R} \]
Constraints:
\[ x + 4y \leq 12 \]
\[ x - y \leq 2 \]
\[ x \geq -4 \]
Objective function:
\[ P = x - y \]
3. Meg is building a bookshelf to display her cookbooks and novels.
   - She has no more than 50 cookbooks and no more than 200 novels.
   - She wants to display at least 2 novels for every cookbook.
   - The cookbook spines are about half an inch wide, and the novel spines are about a quarter of an inch wide.

Meg wants to know how long to make the bookshelf.

The following model represents this situation.

Let $c$ represent the number of cookbooks.
Let $n$ represent the number of novels.
Let $W$ represent the width of the bookshelf.

Restrictions:
$c \leq 50$
$n \leq 200$
$n \geq 2c$

Constraints:
$c \geq 0$
$n \geq 0$
$c \leq 50$
$n \leq 200$

Objective function:
$W = 0.5c + 0.25n$

a) Which point in the feasible region represents the greatest number of books (both cookbooks and novels) that Meg could have? Explain how you know.
b) Can she display the same number of cookbooks as novels? Explain.
c) Which point represents the most cookbooks and the fewest novels?
d) What point represents the number of cookbooks that would require the longest shelf? How long would the shelf have to be?
e) What point represents the number of cookbooks that would require the shortest shelf?
Optimization Problems III: Linear Programming

LEARN ABOUT the Math

In Lesson 6.4, you created an optimization model for the problem below. In Lesson 6.5, you explored the model for optimal solutions.

A toy company manufactures two types of toy vehicles: racing cars and sport-utility vehicles.

- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs $8 to make a racing car and $12 to make a sport-utility vehicle.

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.

Which combinations of the two types of vehicles will result in a minimum and a maximum production cost, and what will these costs be?

**Example 1** Solving an optimization problem by graphing an algebraic model

Ramona’s Solution

- Let \( s \) represent sport-utility vehicles.
- Let \( r \) represent racing cars.
- Restrictions on the variables:
  - \( r \in \mathbb{W}, s \in \mathbb{W} \)
  - Constraints:
    - \( r \geq 0 \)
    - \( s \geq 0 \)
    - \( r \leq 40 \)
    - \( s \leq 60 \)
    - \( r + s \geq 70 \)

Let \( C \) represent the total production cost. Objective function to optimize, at $12 per sport-utility vehicle and $8 per car:
\[
C = 12s + 8r
\]
I graphed the system of inequalities to determine the points at the vertices of the feasible region.

I knew that two of these points represent the solutions that will optimize the objective function.

I conjectured that

- the minimum solution is (30, 40) because this represents the lowest total number of vehicles (70) and there are fewer of the more expensive sport-utility vehicles than racing cars (30 versus 40).
- the maximum solution is (60, 40) because this represents the highest total number of vehicles (100) and there is a greater number of the more expensive sport-utility vehicles than racing cars (60 versus 40).

To check my conjecture, I evaluated the objective function using the coordinates of each vertex of the feasible region. This enabled me to compare the production cost for each solution.

My conjecture was correct.

I verified each optimal solution to make sure it satisfied every constraint in the system. Each inequality statement was valid for both optimal solutions.

The company can minimize the production cost to $680 by making 30 sport-utility vehicles and 40 racing cars and maximize costs to $1040 by making 60 sport-utility vehicles and 40 racing cars.

Verify minimum (30, 40):

<table>
<thead>
<tr>
<th>(s, r) is (30, 40):</th>
<th>(s, r) is (30, 40):</th>
<th>(s, r) is (30, 40):</th>
</tr>
</thead>
<tbody>
<tr>
<td>r \leq 40</td>
<td>s \leq 60</td>
<td>r + s \geq 70</td>
</tr>
<tr>
<td>LS</td>
<td>RS</td>
<td>LS</td>
</tr>
<tr>
<td>40</td>
<td>30</td>
<td>30 + 40</td>
</tr>
<tr>
<td>40 \leq 40</td>
<td>30 \leq 60</td>
<td>70 \geq 70</td>
</tr>
</tbody>
</table>

Verify minimum (60, 40):

<table>
<thead>
<tr>
<th>(s, r) is (60, 40):</th>
<th>(s, r) is (60, 40):</th>
<th>(s, r) is (60, 40):</th>
</tr>
</thead>
<tbody>
<tr>
<td>r \leq 40</td>
<td>s \leq 60</td>
<td>r + s \geq 70</td>
</tr>
<tr>
<td>LS</td>
<td>RS</td>
<td>LS</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
<td>60 + 40</td>
</tr>
<tr>
<td>40 \leq 40</td>
<td>60 \leq 60</td>
<td>100 \geq 70</td>
</tr>
</tbody>
</table>
I began by defining the variables in this situation. Since both \( n \) and \( w \) represent the number of boards needed, they must be whole numbers.

Next, I created the constraints in the problem. I represented these constraints with linear inequalities.

I converted yards to inches using the rate 36 in./1 yd, since the coefficients of the other terms are in inches.

Reflecting

A. How would the solution change if the cost to make a racing car was $12 and the cost to make a sport-utility vehicle was $8? How could you have predicted this?

B. How would the solution change if the cost was $10 for each vehicle? How could you have predicted that?

C. Summarize the steps you can follow when using linear programming to solve an optimization problem.

**APPLY the Math**

**Example 2**

Creating a model for an optimization problem, and solving the problem

L&G Construction is competing for a contract to build a fence.

- The fence will be no longer than 50 yd and will consist of narrow boards that are 6 in. wide and wide boards that are 8 in. wide.
- There must be no fewer than 100 wide boards and no more than 80 narrow boards.
- The narrow boards cost $3.56 each, and the wide boards cost $4.36 each.

Determine the maximum and minimum costs for the lumber to build the fence.

**Aisla’s Solution**

Let \( n \) represent the number of narrow boards. Let \( w \) represent the number of wide boards. Let \( C \) represent the total cost of the lumber.

Restrictions:

\[ n \in \mathbb{W} \text{ and } w \in \mathbb{W} \]

No fewer than 100 wide boards:

\[ w \geq 100 \]

No more than 80 narrow boards:

\[ n \leq 80 \]

The fence is no longer than 50 yd and is made of 6 in. narrow and 8 in. wide boards:

\[ 6n + 8w \leq 50(36) \]

\[ 6n + 8w \leq 1800 \]
**Optimization Model**

Constraints:
\[ n \geq 0 \]
\[ w \geq 0 \]
\[ w \geq 100 \]
\[ n \leq 80 \]
\[ 6n + 8w \leq 1800 \]

Objective function to optimize at $3.56 per narrow board and $4.36 per wide board:
\[ C = 3.56n + 4.36w \]

I created the objective function that represents the relationship between the cost of the two widths of boards and the total cost of the lumber.

I graphed in the first quadrant because the domain and range are restricted to whole numbers. I decided to make the number of narrow boards \((n)\) the independent variable.

I identified the coordinates of the vertices of the feasible region.

I made these conjectures:
- Point \((80, 165)\) will result in the maximum cost since it is farthest from both axes, which means it has large coordinate values.
- Point \((0, 100)\) will result in the minimum cost, since one coordinate is 0 and the other coordinate is only 100.

I substituted the values of \(n\) and \(w\) for all four vertices into the objective function to compare the cost for these solutions.

My conjectures were correct.

- If \((n, w) = (0, 100)\),
  \[ C = 3.56(0) + 4.36(100) \]
  \[ C = 436.00 \]
- If \((n, w) = (80, 100)\),
  \[ C = 3.56(80) + 4.36(100) \]
  \[ C = 720.80 \]
- If \((n, w) = (80, 165)\),
  \[ C = 3.56(80) + 4.36(165) \]
  \[ C = 1004.20 \]
- If \((n, w) = (0, 225)\),
  \[ C = 3.56(0) + 4.36(225) \]
  \[ C = 981.00 \]

(0, 100), or no narrow boards and 100 wide boards, cost the minimum amount: $436.

(80, 165), or 80 narrow boards and 165 wide boards, cost the maximum amount: $1004.20.
Verify minimum (0, 100):

<table>
<thead>
<tr>
<th>(n, w) is (0, 100):</th>
<th>n ≤ 80</th>
<th>w ≥ 100</th>
<th>6n + 8w ≤ 1800</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS</td>
<td>0</td>
<td>100</td>
<td>6(0) + 8(100)</td>
</tr>
<tr>
<td>RS</td>
<td>80</td>
<td>100</td>
<td>1800</td>
</tr>
</tbody>
</table>

Verify maximum (80, 165):

<table>
<thead>
<tr>
<th>(n, w) is (80, 165):</th>
<th>n ≤ 80</th>
<th>w ≥ 100</th>
<th>6n + 8w ≤ 1800</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS</td>
<td>80</td>
<td>165</td>
<td>6(80) + 8(165)</td>
</tr>
<tr>
<td>RS</td>
<td>80</td>
<td>165</td>
<td>1800</td>
</tr>
</tbody>
</table>

I verified both solutions to make sure they satisfied every constraint in the system. Each inequality statement was valid for both optimal solutions.

Your Turn

How would the feasible region change if 80 or more narrow boards had to be used?
In Summary

Key Idea

- To solve an optimization problem using linear programming, begin by creating algebraic and graphical models of the problem (as shown in Lesson 6.4). Then use the objective function to determine which vertex of the feasible region results in the optimal solution.

Need to Know

- The solution to an optimization problem is usually found at one of the vertices of the feasible region.
- To determine the optimal solution to an optimization problem using linear programming, follow these steps:
  
  **Step 1.** Create an algebraic model that includes:
  - a defining statement of the variables used in your model
  - the restrictions on the variables
  - a system of linear inequalities that describes the constraints
  - an objective function that shows how the variables are related to the quantity to be optimized

  **Step 2.** Graph the system of inequalities to determine the coordinates of the vertices of its feasible region.

  **Step 3.** Evaluate the objective function by substituting the values of the coordinates of each vertex.

  **Step 4.** Compare the results and choose the desired solution.

  **Step 5.** Verify that the solution(s) satisfies the constraints of the problem situation.

CHECK Your Understanding

1. Determine the optimal solutions for the system of linear inequalities graphed below, using the objective function $G = 2x + 5y$. 
2. The following model represents an optimization problem. Determine the maximum solution.

**Optimization Model**

Restrictions:
\[ x \in \mathbb{R} \text{ and } y \in \mathbb{R} \]

Constraints:
\[ x \geq 0 \]
\[ y \leq 0 \]
\[ 3y \geq -2x + 3 \]
\[ y \geq 2x - 7 \]

Objective function:
\[ D = -5x + 3y \]

3. Three teams are travelling to a basketball tournament in cars and minivans.
- Each team has no more than 2 coaches and 14 athletes.
- Each car can take 4 team members, and each minivan can take 6 team members.
- No more than 4 minivans and 12 cars are available.

The school wants to know the combination of cars and minivans that will require the maximum number of vehicles. Create and verify a model to represent this situation.

a) Use the optimization model to determine the combination of cars and minivans that will use the maximum number of vehicles.

b) How many team members can travel in the maximum number of vehicles?

4. Ed found spiders and crickets in his storage room.
- There were 20 or fewer spiders and 20 or more crickets.
- There were 45 or fewer crickets and spiders, in total. Spiders have 8 legs, and crickets have 6 legs.

a) What combination of spiders and crickets would have the greatest number of legs?

b) What combination would have the least number of legs?
5. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model
Restrictions:
\( x \in W, y \in W \)

Constraints:
\( x \geq 0 \)
\( y \geq 0 \)
\( x + y \leq 5 \)
\( 2x + y \geq 5 \)

Objective function:
\( K = -x + 2y \)

6. The following model represents an optimization problem. Determine the minimum solution.

Optimization Model
Restrictions:
\( x \in W, y \in W \)

Constraints:
\( x \geq 0 \)
\( y \geq 0 \)
\( 3x + y \geq 15 \)
\( x \leq 10 \)
\( x \leq 9 \)

Objective function:
\( P = 5x + 3y \)

7. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model
Restrictions:
\( m \in \mathbb{R}, s \in \mathbb{R} \)

Constraints:
\( m \geq 0 \)
\( s \geq 0 \)
\( 3m + 4s \leq 24 \)
\( m + s \geq 4 \)

Objective function:
\( T = 1.5m + 4.2s \)
8. A refinery produces oil and gas.
   • At least 2 L of gasoline is produced for each litre of heating oil.
   • The refinery can produce up to 9 million litres of heating oil and 6 million litres of gasoline each day.
   • Gasoline is projected to sell for $1.10 per litre. Heating oil is projected to sell for $1.75 per litre.
   The company needs to determine the daily combination of gas and heating oil that must be produced to maximize revenue. Create a model to determine this combination. What would the revenue be?

   **Optimization Model**
   Let \( g \) represent the number of litres of gasoline.
   Let \( h \) represent the number of litres of heating oil.
   Let \( R \) represent the total revenue from sales.
   Restrictions:
   \( g, h \in \mathbb{R} \)
   Constraints:
   \( g \geq 0 \)
   \( h \geq 0 \)
   \( g \geq 2h \)
   \( g \leq 6,000,000 \)
   \( h \leq 9,000,000 \)
   Objective function to maximize:
   \( R = 1.10g + 1.75h \)

9. Northwest Trail Mix Limited (NTML) is preparing 1 kg bags of nuts to sell.
   • NTML decides to make and sell no fewer than 3000 bags of walnuts, and no more than 5000 bags of almonds.
   • The marketing department has predicted sales of no fewer than 6000 bags altogether.
   • NTML wants to minimize costs.
   The cost per kilogram is shown in the chart.
   a) Write a system of linear inequalities to describe these constraints:
      i) the number of bags of almonds
      ii) the number of bags of walnuts
      iii) the total number of bags to be sold
   b) Describe the restrictions on the domain and range of the variables.
   c) Graph the system of linear inequalities.
   d) Describe the feasible region.
   e) Write the objective function to represent the quantity to be minimized.
   f) Determine the minimum cost for NTML.

10. Choose two optimization models that were developed in the Practising questions in Lesson 6.4. Solve each model.
11. On a flight between Winnipeg and Vancouver, there are business class and economy seats.
• At capacity, the airplane can hold no more than 145 passengers.
• No fewer than 130 economy seats are sold, and no more than 8 business class seats are sold.
• The airline charges $615 for business class seats and $245 for economy seats.

What combination of business class and economy seats will result in the maximum revenue? What will this maximum revenue be?

12. A school is organizing a track and field meet.
• There will be no more than 250 events and no fewer than 100 events to be scheduled.
• The organizers allow 15 min for each track event and 45 min for each field event.
• They are considering different combinations of track and field events.

What are the least and greatest amounts of time they should allow?

13. Sophie has two summer jobs.
• She works no more than a total of 32 h a week. Both jobs allow her to have flexible hours but in whole hours only.
• At one job, Sophie works no less than 12 h and earns $8.75/h.
• At the other job, Sophie works no more than 24 h and earns $9.00/h.

What combination of numbers of hours will allow her to maximize her earnings? What can she expect to earn?

14. A jewellery store sells diamond earrings: small earrings (no more than 1 carat of diamonds) and large earrings (more than 1 carat of diamonds).
• They sell at least four pairs of small earrings for every pair of large earrings.
• They also sell no more than 120 pairs of earrings, in total, per month.
• The small earrings sell for about $800 a pair, and the large earrings sell for about $1500 a pair.

What combination of the two categories of earrings should they try to sell to maximize their revenue? What amount of sales can they expect?
15. A hardware store sells both asphalt shingles and cedar shakes as roofing materials.
   • The store stocks at least eight times as many bundles of asphalt shingles as cedar shakes.
   • They also stock at least 200 bundles of cedar shakes.
   • The storeroom has space for no more than 2000 bundles altogether.
   • Each bundle of cedar shakes takes up 1.5 cubic feet, and each bundle of asphalt shingles takes up 1 cubic foot.
What combination of asphalt shingles and cedar shakes should the store stock to minimize its storage needs and still keep enough roofing materials on hand for sales?

Closing

16. In Lesson 6.4, question 8, you developed a list of questions that could help you create a model for an optimization problem. Develop another list of questions that could help you use this model to solve the problem.

Extending

17. Choose one of these contexts, or create your own context, and write an optimization problem based on this context. Trade problems with a partner, and solve each other’s problem.
   • using a rental van or a moving company to move
   • building a library with hardcover and paperback books
   • watching movies by going to a theatre or by renting them
1. Identify any errors in each graph. Explain your decisions.

   a) \[(x, y) \mid 4 + 2x \leq 3y, x \in \mathbb{R}, y \in \mathbb{R}\]\n   \[(x, y) \mid 5x - 2y > 6, x \in \mathbb{R}, y \in \mathbb{R}\]

   b) \[(x, y) \mid x - 3y \leq 4 - 3y, x \in \mathbb{W}, y \in \mathbb{W}\]
   \[(x, y) \mid 2y > x + 7, x \in \mathbb{W}, y \in \mathbb{W}\]

2. This model could be used to solve an optimization problem. What point in the feasible region would result in the minimum value for the objective function? What point would result in the maximum value? Explain how you know.

3. Ribbon flowers and crepe-paper rosettes are being made as decorations.
   • At least 50 ribbon flowers and no more than 75 rosettes are needed.
   • Altogether, no more than 140 decorations are needed.
   • Each ribbon flower takes 6 min to make, and each rosette takes 9 min to make.

   What combination of ribbon flowers and rosettes will take the least amount of time to make? What is the minimum time needed to make these decorations?

4. A transportation company leases vehicles.
   • It has 10-passenger vans and 16-passenger minibuses to lease.
   • At most, 5 minibuses are available to lease.
   • There are 120 or fewer people to be transported.
   • Each minibus plus a driver costs $730 to lease, and each van plus a driver costs $550.

   What combination of vans and minibuses will allow the transportation company to maximize the value of the leases? What will the maximum value be? How many people can be transported?

**WHAT DO You Think Now?** Revisit What Do You Think? on page 293. How have your answers and explanations changed?
FREQUENTLY ASKED Questions

Q: What do each of these parts describe in the algebraic model of an optimization problem: a system of linear inequalities and an objective function?

A: The system of linear inequalities describes the constraints of the problem (each one represented by a linear inequality) and the restrictions on the variables (the set of numbers that the variables belong to and any limits on the values of the variables). The objective function describes how the quantity that is being optimized relates to the variables.

Q: Once you have modelled an optimization problem, how do you solve it?

A: The feasible region of a graph for a system of linear inequalities represents all the valid solutions to the system. The optimal solution is usually represented by a point at a vertex of the feasible region.

For example, consider this problem and its optimization model:

An office supply store sells no more than 84 packages of lined paper and graph paper, in total, in a day. The store also sells at least six times as much lined paper as graph paper. Lined paper sells for $2.75 and graph paper for $4.25. How many packages of each paper does the store need to sell to maximize revenue?

Optimization Model:

Let \( g \) represent the number of packages of graph paper.
Let \( l \) represent the number of packages of lined paper.
Let \( R \) represent the revenue.

Restrictions:
\[ l \in \mathbb{W}, g \in \mathbb{W} \]

Constraints:
\[ l \geq 0 \]
\[ g \geq 0 \]
\[ l + g \leq 84 \]
\[ l \geq 6g \]

Objective function:
\[ R = 2.75l + 4.25g \]

<table>
<thead>
<tr>
<th>(( g ), ( l ))</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(0, 84)</td>
<td>231</td>
</tr>
<tr>
<td>(12, 72)</td>
<td>249</td>
</tr>
</tbody>
</table>

The solution that results in the greatest revenue, $249, is 72 packages of lined paper and 12 packages of graph paper.
PRACTISING

Lesson 6.1

1. Graph each linear inequality. Justify each boundary and the half plane you shaded.
   a) \( \{(x, y) \mid 6x + y > 12, x \in \mathbb{I}, y \in \mathbb{I}\} \)
   b) \( \{(x, y) \mid 10 + 2y \leq 7x, x \in \mathbb{W}, y \in \mathbb{W}\} \)
   c) \( \{(x, y) \mid -7y \geq 14, x \in \mathbb{R}, y \in \mathbb{R}\} \)

2. Selma and Claudia are working on a project. The project will take, at most, 50 h to complete. They want to know how many hours each will need to work.
   a) Define the variables and their domain and range. Write and graph a linear inequality to represent this problem.
   b) Choose three combinations of hours that Selma and Claudia could work. How could you verify each combination?

Lesson 6.2

3. Paul is making a mixture of raisins and peanuts.
   a) Define the variables and their domain and range. Write a system of linear inequalities to represent this situation.
   b) Graph the system. Describe the solution region.
   c) Choose three points in the solution region, and explain what each point represents.

Lesson 6.3

4. a) Graph the solution set for the following system of linear inequalities:
   \[ \begin{align*}
   2x + y &< 3y \\
   2y - 5x & \leq 10
   \end{align*} \]
   b) How would the graph change if
      i) the domain and range were from the set of integers?
      ii) the domain and range were from the set of whole numbers?
      iii) the inequality signs were reversed?

5. George is replacing the halyards (ropes that lift the sails) and sheets (ropes that control side movement of the sails) on his boat.
   • He wants no more than 50 m of rope for the halyards.
   • He needs no more than 120 m of rope altogether.
   a) What are the restrictions on the variables? How do you know?
   b) Create a graphical model of this situation and use it to choose two possible combinations of lengths of rope.

6. Consider the following graph of a system of linear inequalities.
   a) Determine the linear equation that represents the boundary of each linear inequality. Explain how you determined each equation.
   b) Represent the system of linear inequalities algebraically.
   c) Verify that your system in part b) matches the graph.
   d) What are the restrictions on the variables? How do you know?
Lesson 6.4

7. A pet store specializes in birds.
   - It sells at least three times more male birds than female birds of the same species. The males’ colourful feathers make them more popular.
   - Over the past two weeks, no more than 28 birds, in total, have been sold.
   - Males were sold for $115, and females were sold for $90.

What combinations of sales of male and female birds would have maximized the pet store’s revenue? Create a model of this problem.

8. A zoo has categorized its exhibits as herbivores and carnivores.
   - There are no more than 50 exhibits altogether.
   - No more than 50% of the 50 possible exhibits are herbivores, and no less than 30% are carnivores.
   - A ticket to any herbivore exhibit costs $15, and a ticket to any carnivore exhibit costs $18.

What combinations of herbivore and carnivore exhibits would maximize the zoo’s revenue? Create a model of this problem.

Lesson 6.5

9. The following model represents an optimization problem. What point in the feasible region would result in the minimum value for the objective function? What point would result in the maximum value? Explain how you know.

Optimization Model

Restrictions:
\[ x \in \mathbb{R}, \quad y \in \mathbb{R} \]

Constraints:
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ y - 4x \geq -12 \]
\[ x + 2y \leq 12 \]

Objective function:
\[ B = -2x + y \]

Lesson 6.6

10. The following model represents an optimization problem. Determine the maximum solution.

Optimization Model

Restrictions:
\[ x \in \mathbb{R}, \quad y \in \mathbb{R} \]

Constraints:
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ 3y \geq 5x - 15 \]
\[ 3y + x \leq 3 \]

Objective function:
\[ M = 1.5x + 3.1y \]

11. The stylists in a hair salon cut hair for women and men.
   - The salon books at least four women’s appointments for every man’s appointment.
   - Usually there are 90 or more appointments, in total, during a week.
   - The salon is trying to reduce the number of hours the stylists work.
   - A woman’s cut takes about 75 min, and a man’s cut takes about 30 min.

What combination of women’s and men’s appointments would minimize the number of hours the stylists work? How many hours would this be?
A student council is planning a sandwich fundraiser. The council wants to sell two different sandwiches: a meat sandwich and a cheese sandwich. Drey researched the price of the basic sandwich ingredients: multigrain bread, lettuce, tomatoes, Dijon mustard, mayonnaise, and dill pickles. He found that these ingredients would cost no more than $0.47, in total, per sandwich. Ruth researched the prices of meats and cheeses. She made the following list:

<table>
<thead>
<tr>
<th>Sandwich Ingredient</th>
<th>Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turkey Breast Deli Meat</td>
<td>$2.39/100g</td>
</tr>
<tr>
<td>Honey Ham Deli Meat</td>
<td>$1.79/100g</td>
</tr>
<tr>
<td>Pastrami Beef, 98% fat free, Deli Meat</td>
<td>$2.69/100g</td>
</tr>
<tr>
<td>Tofu Turkey</td>
<td>$1.31/100g</td>
</tr>
<tr>
<td>Cream Havarti Cheese</td>
<td>$1.89/100g</td>
</tr>
<tr>
<td>Aged Swiss Cheese</td>
<td>$2.39/100g</td>
</tr>
<tr>
<td>Monterey Jack Cheese</td>
<td>$1.33/100g</td>
</tr>
<tr>
<td>Vegan Cheese</td>
<td>$1.76/100g</td>
</tr>
</tbody>
</table>

Each sandwich will include the basic ingredients and 100 g of meat or cheese. The council plans to make no more than 360 meat sandwiches and no more than 400 cheese sandwiches. The council expects to sell 600 or more sandwiches.

How can the student council minimize its cost? What will the minimum cost be?

A. Choose a meat and a cheese from Ruth’s list for the sandwiches.
B. Model the problem situation.
C. Determine the minimum cost.
D. Write a recommendation to the student council. Include your reasoning.
Identifying Controversial Issues

While working on your research project you may uncover some issues on which people disagree. To decide on how to present an issue fairly, consider some questions you can ask yourself or others as you carry out your research.

1. What is the issue about?
Identify which type of controversy you have uncovered. Almost all controversy revolves around one or more of the following:
   - Values—What should be? What is best?
   - Information—What is the truth? What is a reasonable interpretation?
   - Concepts—What does this mean? What are the implications?

2. What positions are being taken on the issue?
Determine what is being said and whether there is reasonable support for the claims being made. You can ask questions of yourself and of others as you research to test the acceptability of values claims:
   - Would you like that done to you?
   - Is the claim based on a value that is generally shared?
If the controversy involves information, ask questions about the information being used:
   - Is there adequate information?
   - Are the claims in the information accurate?
If the controversy surrounds concepts, look at the words being used:
   - Are those taking various positions on the issue all using the same meanings of terms?

3. What is being assumed?
Faulty assumptions reduce legitimacy. You can ask:
   - What are the assumptions behind an argument?
   - Is the position based on prejudice or an attitude contrary to universally held human values, such as those set out in the United Nations Declaration of Human Rights?
   - Is the person presenting a position or opinion an insider or an outsider?
Insiders may have information and understanding not available to outsiders; however, they may also have special interests. Outsiders may lack the information or depth of understanding available to insiders; however, they may also be more objective.

4. What are the interests of those taking positions?
Try to determine the motivations of those taking positions on the issue. What are their reasons for taking their positions? The degree to which the parties involved are acting in self-interest could affect the legitimacy of their positions.
Sarah chose the changes in population of the Western provinces and the territories over the last century as her topic. Below, she describes how she identified and dealt with a controversial issue.

**Sarah’s Explanation**

I found that population growth can involve controversial issues. One such issue is discrimination in immigration policy. When researching reasons for population growth in Canada, I discovered that our nation had some controversial immigration policies.

From 1880 to 1885 about 17,000 Chinese labourers helped build the British Columbia section of the trans-Canada railway. They were paid only half the wage of union workers. When the railway was finished, these workers were no longer welcome. The federal government passed the *Chinese Immigration Act* in 1885, putting a head tax of $50 on Chinese immigrants to discourage them from staying in or entering Canada. In 1903 the head tax was raised to $500, which was about two years’ pay at the time. In 1923, Canada passed the *Chinese Exclusion Act*, which stopped Chinese immigration to Canada for nearly a quarter of a century. In 2006, Prime Minister Stephen Harper made a speech in the House of Commons apologizing for these policies.

I decided to do some more research on Canada’s immigration policies, looking at historical viewpoints and the impact of the policies on population growth in the West and North. I will include a discussion about the effects of these policies in my presentation and report.

**Your Turn**

A. Identify the most controversial issue, if any, you have uncovered during your research.

B. Determine the different positions people have on this issue and the supporting arguments they present. If possible, include any supporting data for these different positions.

C. If applicable, include a discussion of this issue in your presentation and report.