The Research Behind *Leaps and Bounds Grades 3 to 8*

Leaps and Bounds toward Math Understanding is a comprehensive resource that provides diagnostic tools and remediation/intervention lessons for students who are struggling in Grades 3 to 8 mathematics in all five strands. The approach used in *Leaps and Bounds* respects what we know about how students learn by sequencing the content in a developmentally appropriate way and providing alternatives in approach that allow for differentiated instruction.

Leaps and Bounds has a solid research foundation that reflects the following:

• developmental learning of mathematics, as determined from the PRIME research;

• recognized common areas of difficulty in mathematics that students have, and research in best instructional practices for addressing these areas of difficulty; and

• current research around how to support students who are struggling in math; in particular, addressing both different learning styles and alternative strategies for learning a mathematical concept.

Developmental Learning in Mathematics

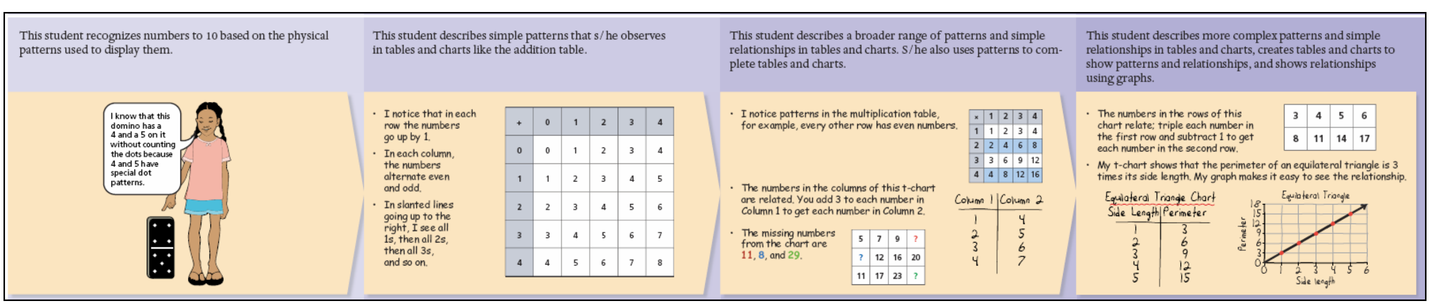
Dr. Marian Small conducted research across Canada to collect data about how elementary students learn mathematics. The research was conducted between 2002 and 2004 in seven provinces with 12,000 students from kindergarten to grade 7. The data collected was the foundation for the developmental maps published under the name of PRIME (Professional Resources and Instruction for Math Educators).

> *PRIME is a Canadian research-based professional learning initiative designed to assist teachers, administrators and district personnel to improve elementary school mathematics instruction and learning.*

Developmental maps were created for the five strands. The maps describe the phases that students travel through as they learn mathematics, and indicate key behaviors that students exhibit at each phase. The maps also reflect the curriculum that is taught in elementary schools across Canada.

There are eight maps altogether: Number, Operations, Patterns, Algebra, Measurement, Geometry, Data Management, and Probability. Each map has two versions, a Phases and Indicators Map and a Visual Overview Map (see below). The maps group the key student behaviours by key concepts or big ideas. The Phases and Indicators Map describes the behaviors as key indicators and the Visual Overview Maps show what those indicators look like. Below is the Phase and Indicators map and related Visual Overview Map for one of the big concepts or big ideas in Patterns and Algebra, Concept 4: Data can be arranged to highlight patterns and relationships.

Identifying and describing patterns			
6 Identifies numbers to 10 based on physical configurations (e.g., without counting, recognizes that 4 dots in a square arrangement is 4).	14 Describes simple patterns in tables and charts.	11 Informally describes a broader range of patterns in tables and charts (e.g., the numbers in the 2nd column increase by 4 each time). 12 Informally describes simple relationships in tables and charts (e.g., multiply each number in the 1st column by 4 to get each related number in the 2nd column).	10 Describes more complex patterns in tables and charts using words. 11 Describes simple relationships in tables and charts using words (e.g., as the number of tricycles increase by 1, the number of wheels increase by 3) or simple sentences (e.g., Column 1 × 2 + 1 = Column 2).
		Creating patterns	
		13 Demonstrates an understanding of the patterns in tables and charts by completing the table or chart.	12 Creates charts and tables to highlight patterns and relationships. 13 Represents and describes some patterns and relationships using graphs.



The maps clearly describe and show how a student develops in his or her understanding of patterns and relationships across four phases.

Note that the research indicated that these phases are not strictly tied to specific grade levels; for example, although most students in Phase 1 are in the early primary grades, there are students in the later elementary grades that are still in Phase 1 and students in the primary grades that are in Phase 2 or 3. The maps help teachers determine what the next steps are for each student, regardless of grade level.

Developmental Learning and Leaps and Bounds

The information provided by the PRIME research and the resulting developmental maps was invaluable in the creation of *Leaps and Bounds*. Carefully sequenced interventions were created based on what we know about development in mathematics and the mathematics curriculum. For example, in Geometry, we know from the research that students develop in their ability to work with representations of 3-D shapes, from working with more concrete models, such as geometric solids and clay models, to more abstract models such as skeletons and nets. So, in *Leaps and Bounds Grades 5/6*, three different intervention pathways have been created to reflect this.

Pathway 1: Modelling With Nets

Pathway 2: Modelling With Skeletons

Pathway 3: Modelling With Solid Shapes

These pathways re-teach the critical concepts behind 3-D representations. A diagnostic tool that highlights the common difficulties that students have is provided to allow a teacher to determine which pathway is required for an individual student. Although a student who is significantly behind might be assigned Pathway 3, it is not always necessary for the student to then complete Pathway 2 and then Pathway 1. After completing Pathway 3, the teacher can reassign the appropriate diagnostic questions from the tool to ascertain whether more intervention is required.

> *Leaps and Bounds allows teachers to re-teach the critical concepts behind each topic in a developmentally appropriate way.*

Common Areas of Difficulty in Mathematics

Most teachers know from experience where students tend to have difficulty in mathematics. As well, there is ample research on this (see the References). And, one of the interesting byproducts of the PRIME research was, while trying to determine how students learn mathematics, common areas of difficulty surfaced, thus providing another source of information about common areas of difficulty.

Certainly an understanding of developmental learning in math can assist in remediating these common difficulties, as some of these arise from students being introduced to concepts before they are ready for them or because students are introduced to concepts in a developmentally inappropriate way. However, we know from the research that there are also specific strategies we can use to target some of the difficulties.

Common Areas of Difficulty and Leaps and Bounds

Simply by creating pathways that reflect what we know about how students develop in their understanding of mathematics will go a long way to help students. However, *Leaps and Bounds* also uses what we know about where and why students struggle in mathematics and how to help them, as a basis to develop the different intervention pathways. So, the pathways not only allow students an opportunity to re-examine concepts in a conceptually meaningful way but also include tasks and questions that target common areas of difficulty. As well, in the *Leaps and Bounds* Teacher Resource, teachers are provided with a list of what the common areas of difficulty are. They are also provided with key behaviours to look for and key questions to ask as students work, which focus on the critical concepts and the common areas of difficulty.

For example, the Teachers Resource for *Leaps and Bounds Grade 3/4*, Topic: Skip Counting, provides the following information in the front matter to the topic:

Why might a student struggle with skip counting?

Skip counting is fundamental to representing numbers and comparing them, particularly on number lines. Many students struggle with skip counting because it requires attention to patterns in the place value system that are not always clear to students.

- Students may struggle over transitions, where more than one digit changes (e.g., going from 108 to 110, or 95 to 100, or 375 to 400).
- Students may have difficulty when not beginning at the start (e.g., starting at 35 instead of starting at 5 when skip counting by 5s).
- Students might struggle when counting backwards.

Sometimes these problems are alleviated with experience. Frequently students are exposed to only limited types of skip-counting situations.

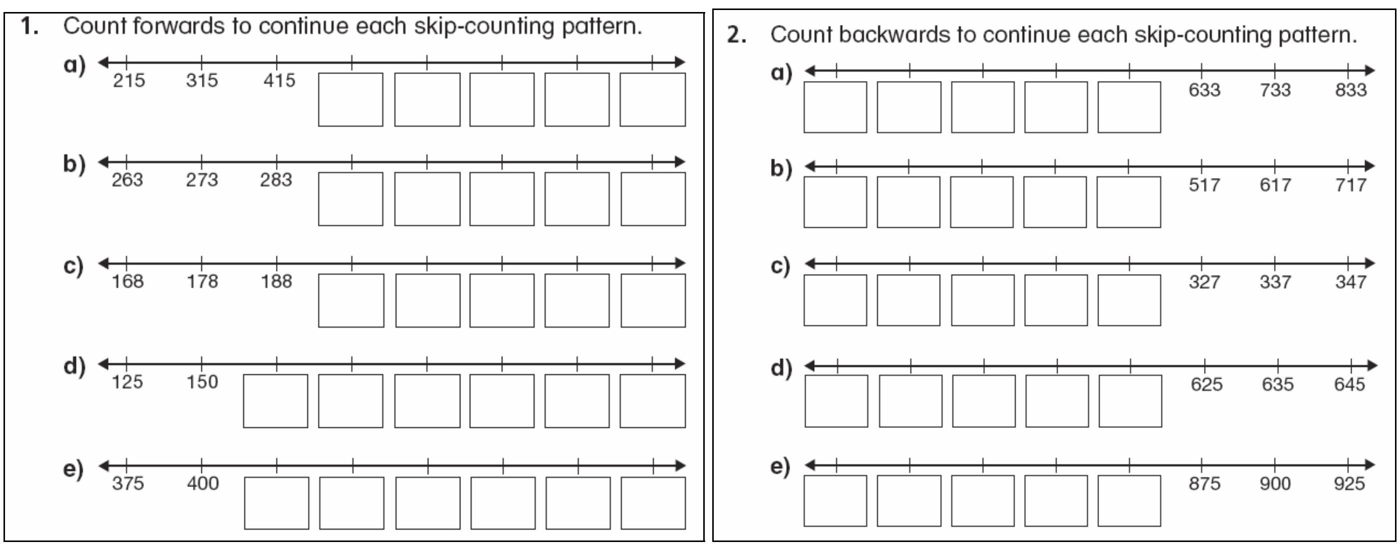
In each of the three intervention pathways for the Skip Counting topic (Skip Counting to 1000, Skip Counting to 100, and Skip Counting to 10), there are opportunities for students to work on questions and tasks that require students to

• make transitions where more than one digit changes,

• skip count from different starting numbers, and

• skip count backwards

For example, here are two questions from the Skip Counting to 1000 intervention pathway:



In the teachers' resource for this pathway, teachers are provided with these key observational indicators:

Observe whether students

- recognize which digits change and why, when counting forwards and backwards by 10s or 100s (Questions 1, 2, 3, 4, 6)
- understand when the hundreds digit changes as they cycle through the endings of 25, 50, 75, and 00 when counting by 25s (Questions 1, 2, 4)
- relate skip counting to adding and subtracting (Question 5)

Knowing where students struggle and why, what to watch and what questions to ask, prepares teachers for assisting students while they work on tasks that are designed to re-teach a topic while focusing on the critical concepts and common areas of difficulty.

Current Research on Supporting Struggling Students

Current research on supporting struggling students suggests the need to incorporate the following in any remediation program:

- Differentiated Instruction
- Conceptually-based explicit instruction
- Visual representation
- Meaningful practice
- Scaffolding
- Math Discussion

Differentiated Instruction and Leaps and Bounds

To differentiate instruction for any student or group of students, both the content that we want students to learn and the strategies or approach used to teach the content can be individualized. *Leaps and Bounds* does both.

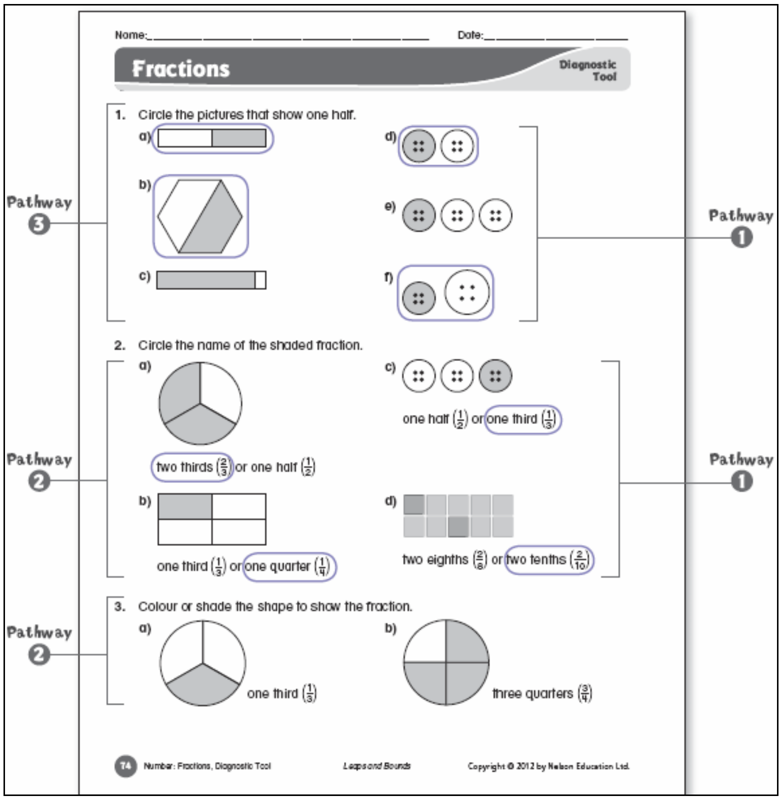
Individualizing Content

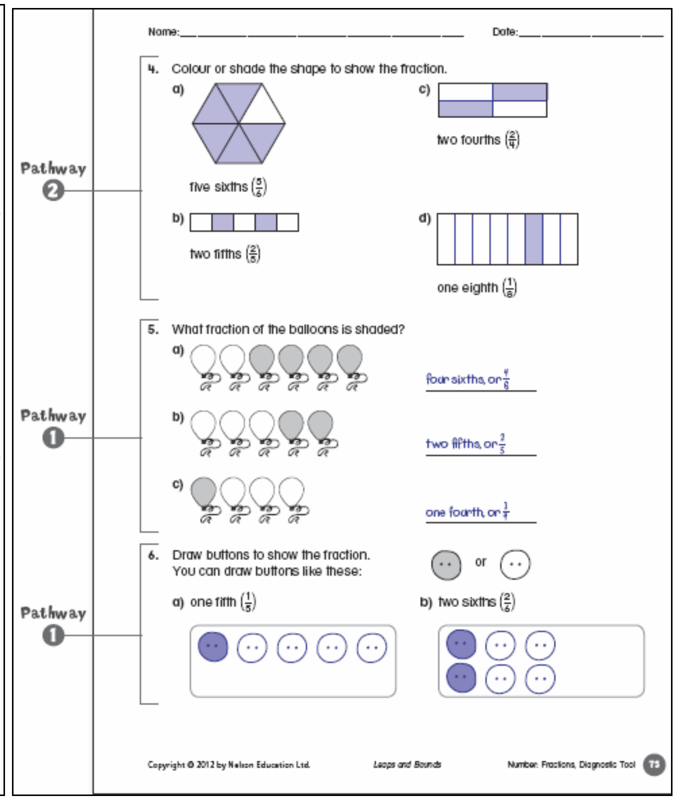
As described earlier, each topic in each strand provides multiple intervention pathways for students to follow, depending on their individual needs. In the teachers' resource for each topic, a diagnostic tool is provided that teachers can administer to students to help decide which pathway is most suitable for the student or group of students. In this way, the content that students are exposed to can be individualized.

For example, in *Leaps and Bounds Grades 3/4*, Topic: Fractions, there is a diagnostic tool that has been crafted to determine what content students are struggling with. We know from the research (both the PRIME research and research on common areas of difficulty), that learning about fractions of a set requires more sophistication than fractions of a region and that the concept of a half is usually a student's first exposure to the topic of fractions. That is why there are three pathways provided (Fractions as Parts of Sets, Fractions as Parts of Regions, and Halves). To determine which pathway is most suitable, the questions on the Tool have been organized such that, if students have difficulty with certain questions and not others, it indicates a specific pathway (as shown in the chart below).

Diagnostic Tool: Fractions

Use the diagnostic tool to determine the most suitable intervention pathway for fractions. Provide Diagnostic Tool: Fractions, Teacher's Resource pages 74 and 75, and have students complete it in writing or orally.





Intervention Pathways

The purpose of the intervention pathways is to help students work with proper fractions (fractions less than 1) as parts of sets *and* as parts of wholes. The focus is to prepare them for working with a broader range of fractions.

There are 3 pathways:

- Pathway 1: Fractions as Parts of Sets
- Pathway 2: Fractions as Parts of Wholes
- Pathway 3: Halves

Use the chart below (or the Key to Pathways on Teacher's Resource pages 76 and 77) to determine which pathway is most suitable for each student or group of students.

Diagnostic Tool Results	Intervention Pathway
If students struggle with Questions 1d–f, 2c–d, 5, 6	use Pathway 1: Fractions as Parts of Sets (*Teacher's Resource pages 78–79* *Student Resource pages 68–72*)
If students struggle with Questions 2a–b, 3, 4	use Pathway 2: Fractions as Parts of Wholes (*Teacher's Resource pages 80–81* *Student Resource pages 73–77*)
If students struggle with Question 1	use Pathway 3: Halves (*Teacher's Resource pages 82–83* *Student Resource pages 78–81*)

Individualizing Instructional Approaches

Once a teacher has determined which pathway is most suitable, the teacher has another choice to make, that is, whether they use a more open approach (the Open-Ended Intervention) or a guided approach (the Guided Intervention). In this way, the instructional approach can also been individualized or differentiated. The two approaches or types of intervention cover the same content but in different ways:

• The Open interventions typically provide a brief introduction with minimal instruction, often in a context, followed by a problem to solve or task that has multiple possible solutions. It allows those students who prefer exploring in their own way more opportunity to do this. Some students who struggle might struggle less if given the opportunity to make their own way through a topic.

• The Guided interventions begin with an instructional section, following by Try These questions which are sequenced (see Scaffolding) to guide the student through the content in a more deliberate way. This suits students who prefer direction, although choices are still provided in terms of how to approach a concept even in this more guided approach.

The teacher can choose the intervention that is more suitable for the student's needs or style and most appropriate for his or her specific learning situation. Teachers might also use both interventions, in either order.

Conceptually-based Explicit Instruction and Leaps and Bounds

In Gersten et al's research, explicit instruction is shown to improve math achievement in struggling learners. Research also indicates that

> "*… providing a mix of direct instruction of new skills and concepts, guided practice, opportunities for complex thinking and problem solving and time for discussion is even more important for the struggling students than for students in general.*" (Elizabeth G Shellard, p. 41).

Explicit Instruction

Leaps and Bounds provides detailed, explicit instructional approaches for each pathway. In the Guided interventions, students are guided through the content in a clear and explicit way in the instructional section. This is followed by carefully sequenced Try These questions which provide opportunities to apply and practice what they have learned. In the Open interventions, although the student task is open, the teachers' resource provides observational prompts for the teacher to watch for and questions for the teacher to ask to ensure the main concepts are addressed through teacher questioning.

Conceptually-Based Instruction

In *Leaps and Bounds*, the guided and open interventions for each pathway provide tasks and questions which encourage students to work conceptually, use complex thinking, and solve problems.

For example, below on the left is an example of an Open intervention task from *Leaps and Bounds 3/4* for the topic Patterns. On the right is one question selected from the Guided intervention from the same pathway. Both the Open and Guided task deal with the concept of repeating patterns and require complex thinking.

Create 3 **repeating patterns** using objects. For each pattern:
- Use at least 3 items in the part that repeats.
- Show at least 3 repetitions.
- Draw or sketch your pattern.
- Describe the pattern.
- Circle the part that repeats. (This is the **pattern core.**)

Compare the 2 patterns. How are they the same? How are they different?

a) 1, 2, 1, 2, 1, 2, 1, 2, … and ◯ ☺ ◯ ☺ ◯ ☺ ◯ ☺ …

same: ____________________

different: ____________________

b) 2, 8, 3, 7, 2, 8, 3, 7, 2, 8, 3, 7, … and 3, 9, 4, 8, 3, 9, 4, 8, 3, 9, 4, 8, 3, 9, …

same: ____________________

different: ____________________

Visual Representations and Leaps and Bounds

Everyone benefits when visual models are used in the classroom. They make it easier to both teach and learn, as they help explain and make sense of the abstract concepts in math. They also provide students with images that they can call upon later to help them use visualization when solving mathematical problems and explain their thinking.

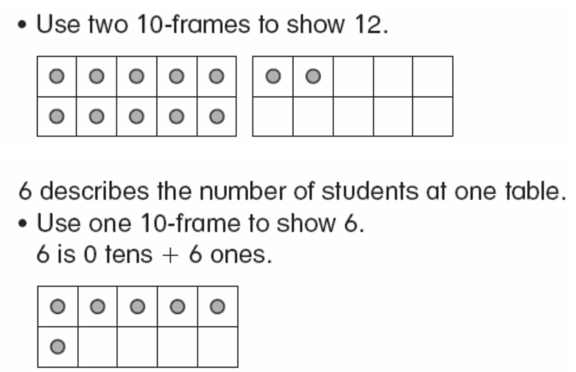
> "*Students with learning disabilities and special needs are predominantly visual learners and benefit when materials are designed to respond to their preferred learning style*." (DeVincentis, p. 5)

In *Leaps and Bounds*, there is a variety of visual representations used to help students "see", understand, and remember the math. Visual representations such as ten frames, number lines, place value charts, labelled diagrams and illustrations are provided to represent abstract concepts, in addition to engaging and appealing to students. Students are also encouraged to use these models when they solve problems and explain their thinking.

For example, here is an array of selected visuals used in all five strands of *Leaps and Bounds 3/4*:

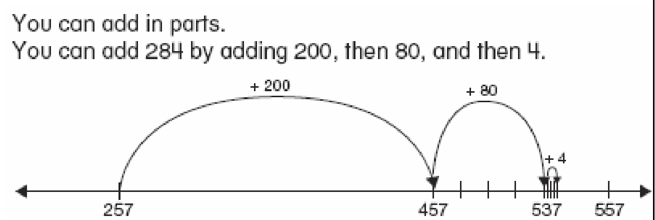
NUMBER

Topic: Representing Whole Numbers



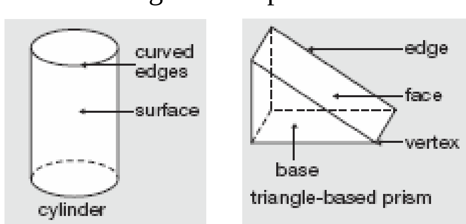
OPERATIONS

Topic: Adding Whole Numbers



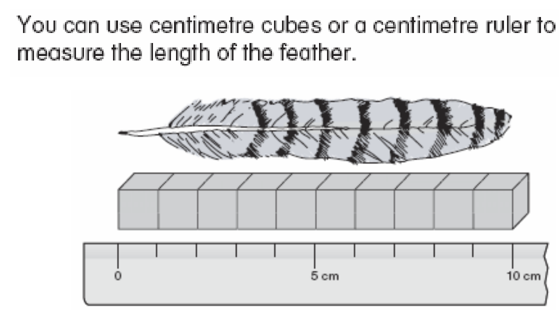
GEOMETRY

Topic: Describing 3-D Shapes



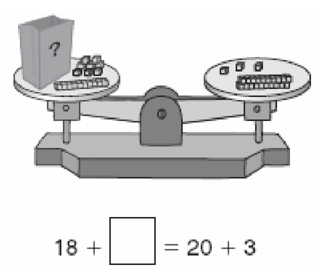
MEASUREMENT

Topic: Length



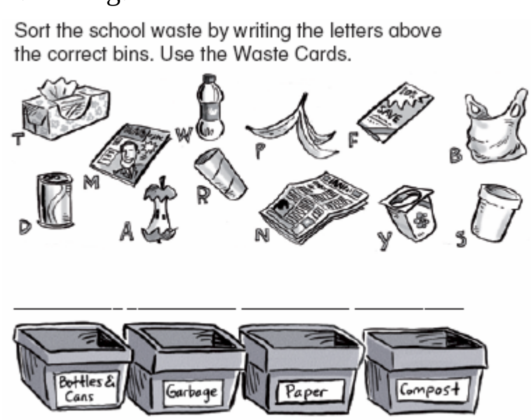
PATTERNS AND ALGEBRA

Topic: Equality



DATA MANAGEMENT

Topic: Sorting: More Than One Attribute



Meaningful Practice and Leaps and Bounds

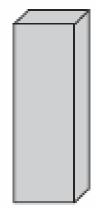
Meaningful practice is a balance of conceptual and procedural work that doesn't lose sight of the big ideas of mathematics. In *Leaps and Bounds*, the **Try These** questions in the Guided interventions provide opportunities for students to apply and practice what they have learned in the instructional section in a meaningful way, using carefully sequenced questions (straightforward to more complex and challenging), a variety of visual representations, and a variety of contexts (both real world and mathematical contexts) so that students can more easily generalize their learning.

The following question from the **Try These** questions for the Topic: Using Non-Standard Units provides a meaningful real world context for students to apply and practice what they have learned about measuring length using nonstandard units. The question uses a different real world context and a different nonstandard unit than was used in the instructional section so students can practise what they have learned in a new situation.

A bookshelf has the space to hold a book that is 8 light green rods tall. Find a book that would fit in this shelf. Find a book that is too tall for this shelf. How many light green rods tall is each book?

Book that fits: about _______ light green rods

Book that is too tall: about _______ light green rods



light green rod

The Open interventions in *Leaps and Bounds* provide meaningful practice by allowing ample opportunity for students to practise the problem solving process. Often the student solves a number of similar problems to increase their opportunity to practise a particular concept. For example, the following task is from the Open Intervention for the Topic Reading a Clock.

Write or draw as many times as you can for this description:

The 2 hands of the clock are fairly close together.

Include only the times where the minute hand points right at a number on the clock.

Scaffolding and Leaps and Bounds

Scaffolding is used throughout *Leaps and Bounds* to prepare students and help them better understand what they are asked to do. For example,

• Perhaps the most effective scaffolding strategy used in *Leaps and Bounds* is the "Before" activity in the teacher's resource. This is a series of questions provided for each intervention that teachers can use to prepare students for what they will be doing in the student resource. For example, the following series of questions will help prepare students for the Open Intervention for the topic Concrete and Picture graphs:

Before Using the Open-Ended Intervention

Use a graphing mat (BLM 35) or make one using a large sheet of chart paper. Create a concrete graph by placing 3 different types of pattern blocks on the mat.

- Is it easy to tell which kind of block we have the most of? Why? (e.g., *yes, because they are lined up and the longest line shows the most blocks*)
- Why is it important to line up the blocks so they match? (e.g., *You want to be able to compare the different lines of blocks to each other.*)
- Why might it be useful to compare the number of blocks in 2 or more categories? (e.g., *You may want to know which category has the most or the least number of blocks.*)

• Marginal Definition boxes and Remember boxes are also used in the student resource to remind students of key terms and concepts that they need to know.

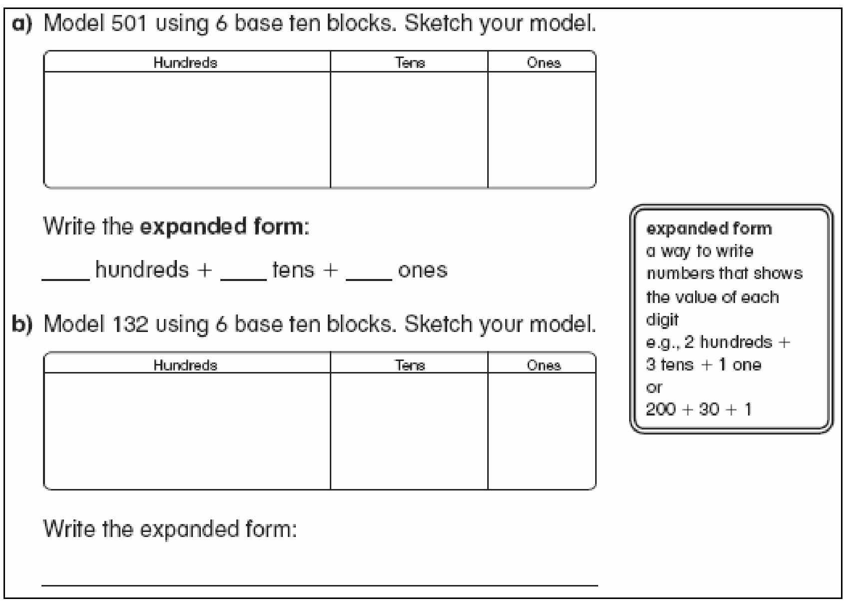
• Another very important strategy is the way the pathways within each topic have been designed to be parallel. So, a student who is successful with Pathway 2 will be prepared to work on Pathway 1, if the teacher feels that the student needs more intervention.

• One simple scaffolding strategy used in all the Try These questions of the Guided interventions is to organize the questions from simple to more challenging.

• Another familiar strategy used is breaking complex questions or problems into logical sequential parts (either using parts a), b), and so on or using bullets) that help students move towards the solution in steps.

• Yet another very simple but helpful strategy used in *Leaps and Bounds* is providing an answer space or number of answer lines that reflects the amount of space or writing the student might need to do to answer the answer.

In the example below, several strategies are used. Part a) provides a an empty place value mat for students to draw in and a fill-in-the-blank answer line for the expanded form of the number which will assist students when they answer part b). As well, the definition in the margin is situated such that, if students have forgotten or are feeling unsure about what expanded form is, they have a definition and model to follow (note that there is also an illustrated glossary of math terms at the back of each student book).


a) Model 501 using 6 base ten blocks. Sketch your model.

Hundreds	Tens	Ones

Write the **expanded form:**

____ hundreds + ____ tens + ____ ones

b) Model 132 using 6 base ten blocks. Sketch your model.

Hundreds	Tens	Ones

Write the expanded form:

____

expanded form
a way to write numbers that shows the value of each digit
e.g., 2 hundreds + 3 tens + 1 one
or
200 + 30 + 1

Below you will see a sampling of strategies used throughout *Leaps and Bounds* to provide scaffolding for students.

Three answer blanks are provided to prompt students to write three numbers. Start at 4 and skip count by 2s. Write 3 numbers you did *not* say. _______ _______ _______	Students are provided with a model and a structure in the form of addition sentence blanks to write their answer. A number added to itself is almost 800. ■■■ + ■■■ = almost 800 What could the number be, and what is the sum? _______ + _______ = _______
Students are provided with a chart to help organize their thinking and their answer.	In this task from an Open intervention, a Remember box in the margin is provided to support students who might be unsure of how to proceed.

Choose an event you might do that does not take a long time.
Record it in the chart below.

Event 1	Estimated	Actual
number of turns of a small timer		
number of turns of a large timer		
number you can count to		
number of times you can empty and refill the pencil case		

Draw your own picture that shows other fractions of sets.
Use big fractions and little ones.

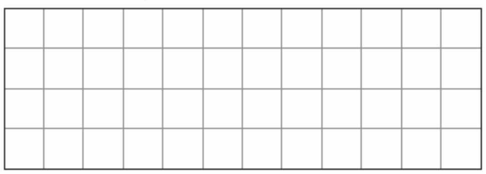
Remember

- A fraction tells about a part of a whole.
- The whole can be a set (group) of objects.
- You might use these fraction words:

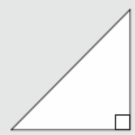
half ●○
third ●○○
fourth ○○●○
fifth ○○●○○

A grid is provided with this question to make it easier for the student to answer without having to find grid paper and answer the question on a separate piece of paper. The illustrated Remember box provides additional support.

Draw 3 polygons that have a square corner.
Name each polygon.



Remember



Right angle or square corner

Students can see that they not only have to provide an estimate but there is an answer line for them to explain their thinking.

Estimate each difference. Explain your thinking.

a) 307 – 297 is about ________.

____

Many of these scaffolding examples are designed to help struggling students who are predominantly visuals learners by providing visual clues.

Math Discussion and Leaps and Bounds

"*Discussing math problems and solutions as a class helps student develop mathematics skills and understanding. For students struggling with math, such discussions can help by providing a window into alternative solution methods or having problems and solutions expressed in student language*." (Shellard, p. 42)

Because *Leaps and Bounds* is a supplementary remediation/intervention resource, most of the mathematical discussions will occur between students working in pairs or in small groups. There will also be valuable discussions between the teacher and the student. Of course, you cannot have discussions unless the tasks and questions are open enough for there to be differences in opinion about how to approach the problem or about what the solution or answer is. The Open interventions in *Leaps and Bounds* are designed to be open enough to make discussion not only inevitable but valuable. The Guided interventions also include some open questions that invite discussion about the bigger ideas behind the lesson. As well, in the teacher's resource for each intervention, a series of Consolidating and Reflecting questions are provided along with sample responses. These are designed to prompt a rich discussion around what the students have been doing. For example,

Here are two sample Open interventions for the Topic: Comparing and Ordering Numbers that will likely result in a rich discussion about what the task is asking them to do, what the word "usually" really means, how best to answer the question, how many examples should be provided,	Pathway 2 Jeff says that a two-digit number with a 3 in it is *usually* less than a two-digit number with a 9 in it. Do you agree? Explain your answer. Pathway 3

whether some answers are better than others, how to organize the answer, and so on.	Two numbers are farther apart than 1 and 9. What are the two numbers? Which is greater? List as many answers as you can. Explain your answers. 

This is the final closing question in the Guided intervention for the Topic: Capacity: Non-standard units. It is open enough to allow for some rich discussion about the limitations of using nonstandard units, an important concept in measurement. The FYI (For Your Information) in the margin helps scaffold the question for students.	Suppose 10 containers of chocolate milk were ordered for a picnic. Do you know for sure how much milk that is? Explain your thinking. __________ __________ __________ **FYI** If you measure in litres, then other people will understand how much you mean. If you measure in "pails," they might not.

Below is the set of Consolidating and Reflecting questions provided for the Open intervention for Mental Math.

Consolidating and Reflecting

Ensure understanding by asking questions based on students' work:

- Why did you change 32 + 59 to 32 + 60? (e.g., *I knew that 59 is close to 60, and I could do 32 + 60 in my head by putting 6 extra tens with 32. Then I could subtract 1.*)
- Would you use the same "friendly" numbers to add 26 + 74 and 32 + 79? Explain your thinking. (e.g., *I wouldn't. I know that 1 quarter and 3 quarters makes a dollar, so I do 26 + 74 as 25 + 75 and then I add 1 and take away 1. But for 32 + 79, I'd probably do 32 + 80 and then take away 1.*)
- What was different about calculating 52 − 39 compared with 84 − 46 using mental math? (e.g., *For 52 − 39, I subtracted too much (40) and then added back 1, but for 84 − 46, I subtracted too little (44) and then subtracted 2 more.*)
- I noticed you added 47 − 18 to your second group. Why do you think it belonged to that group? (e.g., *because I subtracted a little extra and added some back*)

In Summary

Leaps and Bounds Toward Math Understanding is an intervention resource designed to help students who are struggling in Grades 3 to 8 mathematics. It is based on current research about developmental learning in mathematics, recognized common areas of difficulty and remedial strategies, and how to support students who are struggling in math through differentiated instruction, conceptually-based explicit instruction, visual represenation, meaningful practice, scaffolding, and math discussion.

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