Chapter 3

DERIVATIVES AND THEIR APPLICATIONS

We live in a world that is always in flux. Sir Isaac Newton’s name for calculus was “the method of fluxions.” He recognized in the seventeenth century, as you probably recognize today, that understanding change is important. Newton was what we might call a “mathematical physicist.” He developed his method of fluxions as a means to better understand the natural world, including motion and gravity. But change is not limited to the natural world, and since Newton’s time, the use of calculus has spread to include applications in the social sciences. Psychology, business, and economics are just a few of the areas in which calculus continues to be an effective problem-solving tool. As we shall see in this chapter, anywhere that functions can be used as models, the derivative is certain to be meaningful and useful.

CHAPTER EXPECTATIONS
In this chapter, you will
• make connections between the concept of motion and the concept of the derivative, Section 3.1
• solve problems involving rates of change, Section 3.1
• determine second derivatives, Section 3.1
• determine the extreme values of a function, Section 3.2
• solve problems by applying a mathematical model and the derivative to determine, interpret, and communicate the mathematical results, Section 3.3
• solve problems by determining the maximum and minimum values of a mathematical model, Section 3.3, 3.4, Career Link
Review of Prerequisite Skills

Now that you have developed your understanding of derivatives and differentiation techniques used in Chapter 2, we will consider a variety of applications of derivatives. The following skills will help you in your work in this chapter:

- graphing polynomial and simple rational functions
- working with circles in standard position
- solving polynomial equations
- finding the equations of tangents and normals
- familiarity with the following formulas:
  - Circle: \( C = 2\pi r, A = \pi r^2 \)
  - Right Circular Cylinder: \( SA = 2\pi rh + 2\pi r^2, V = \pi r^2h \)

Exercise

1. Sketch the graph of each function.
   a. \( 2x + 3y - 6 = 0 \)
   b. \( 3x - 4y = 12 \)
   c. \( y = \sqrt{x} \)
   d. \( y = \sqrt{x - 2} \)
   e. \( y = x^2 - 4 \)
   f. \( y = -x^2 + 9 \)

2. Solve each of the following equations, \( x, t \in \mathbb{R} \).
   a. \( 3(x - 2) + 2(x - 1) - 6 = 0 \)
   b. \( \frac{1}{3}(x - 2) + \frac{2}{5}(x + 3) = \frac{x - 5}{2} \)
   c. \( t^2 - 4t + 3 = 0 \)
   d. \( 2t^2 - 5t - 3 = 0 \)
   e. \( \frac{6}{t} + \frac{t}{2} = 4 \)
   f. \( x^3 + 2x^2 - 3x = 0 \)
   g. \( x^3 - 8x^2 + 16x = 0 \)
   h. \( 4t^3 + 12t^2 - t - 3 = 0 \)
   i. \( 4t^4 - 13t^2 + 9 = 0 \)

3. Solve each inequality, \( x \in \mathbb{R} \).
   a. \( 3x - 2 > 7 \)
   b. \( x(x - 3) > 0 \)
   c. \( -x^2 + 4x > 0 \)
4. Determine the area of the figure described. Leave your answers in terms of $\pi$, where applicable.
   a. Square: perimeter 20 cm
   b. Rectangle: length 8 cm, width 6 cm
   c. Circle: radius 7 cm
   d. Circle: circumference $12\pi$ cm

5. Two measures of a right circular cylinder are given. Calculate the two remaining measures.

<table>
<thead>
<tr>
<th>Radius $r$</th>
<th>Height $h$</th>
<th>Surface Area $S = 2\pi rh + 2\pi r^2$</th>
<th>Volume $V = \pi r^2 h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 4 cm</td>
<td>3 cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. 4 cm</td>
<td></td>
<td></td>
<td>$96\pi$ cm$^3$</td>
</tr>
<tr>
<td>c.</td>
<td>6 cm</td>
<td></td>
<td>$216\pi$ cm$^3$</td>
</tr>
<tr>
<td>d. 5 cm</td>
<td></td>
<td></td>
<td>$120\pi$ cm$^2$</td>
</tr>
</tbody>
</table>

6. Calculate each total surface area and volume for cubes with the following dimensions.
   a. 3 cm
   b. $\sqrt{5}$ cm
   c. $2\sqrt{3}$ cm
   d. $2k$ cm

7. Express each of the following sets of numbers using interval notation.
   a. $\{x \in \mathbb{R} \mid x > 3\}$
   b. $\{x \in \mathbb{R} \mid x \leq -2\}$
   c. $\{x \in \mathbb{R} \mid x < 0\}$
   d. $\{x \in \mathbb{R} \mid x \geq -5\}$
   e. $\{x \in \mathbb{R} \mid -2 < x \leq 8\}$
   f. $\{x \in \mathbb{R} \mid -4 < x < 4\}$

8. Express each of the following intervals using set notation where $x \in \mathbb{R}$.
   a. $(5, \infty)$
   b. $(-\infty, 1]$    
   c. $(-\infty, \infty)$
   d. $[-10, 12]$
   e. $(-1, 3)$
   f. $[2, 20)$

9. Use graphing technology to graph each of the following functions and determine its maximum and/or minimum value.
   a. $f(x) = x^2 - 5$
   b. $f(x) = -x^2 - 10x$
   c. $f(x) = 3x^2 - 30x + 82$
   d. $f(x) = |x| - 1$
   e. $f(x) = 3\sin x + 2$
   f. $f(x) = -2\cos 2x - 5$
CHAPTER 3: MAXIMIZING PROFITS

We live in a world that demands we determine the best, the worst, the maximum, and the minimum. Through mathematical modelling, calculus can be used to establish optimum operating conditions for processes that seem to have competing variables. For example, minimizing transportation costs for a delivery truck would seem to require the driver to travel as fast as possible to reduce hourly wages. Higher rates of speed, however, increase gas consumption. With calculus, an optimal speed can be established that minimizes the total cost of driving the delivery vehicle considering both gas consumption and hourly wages. In this chapter, calculus tools will be utilized in realistic contexts to solve optimization problems—from business applications (e.g., minimizing cost) to psychology (e.g., maximizing learning).

Case Study—Entrepreneurship

In the last 10 years, the Canadian economy has seen a dramatic increase in the number of small businesses. An ability to use graphs to interpret the marginal profit (a calculus concept) will help an entrepreneur make good business decisions.

A person with an old family recipe for gourmet chocolates decides to open her own business. Her weekly total revenue ($TR$) and total cost ($TC$) curves are plotted on the set of axes shown.

DISCUSSION QUESTIONS

Make a rough sketch of the graph in your notes and answer the following questions.

1. What sales interval would keep the company profitable? What do we call these values?

2. Superimpose the total profit ($TP$) curve over the $TR$ and $TC$ curves. What would the sales level have to be to obtain maximum profits? Estimate the slopes on the $TR$ and $TC$ curves at this level of sales. Should they be the same? Why or why not?

3. On a separate set of axes, draw a rough sketch of the marginal profit ($MP = \frac{dTP}{dx}$), the extra profit earned by selling one more box of chocolates. What can you say about the marginal profit as the level of sales progresses from just less than the maximum, to the maximum, to just above the maximum? Does this make sense? Explain.
Section 3.1—Higher-Order Derivatives, Velocity, and Acceleration

Derivatives arise in the study of motion. The velocity of a car is the rate of change of displacement at a specific point in time. Up to this point, we have developed the rules of differentiation and learned how to interpret the derivative at a point on a curve. We can now extend the applications of differentiation to higher-order derivatives. This will allow us to discuss the application of the first and second derivatives to rates of change as an object moves in a straight line, either vertically or horizontally, such as a space shuttle taking off into space or a car moving along a straight section of road.

Higher-Order Derivatives
The function $y = f(x)$ has a first derivative $y = f'(x)$. The second derivative of $y = f(x)$ is the derivative of $y = f'(x)$.

The derivative of $f(x) = 10x^4$ with respect to $x$ is $f'(x) = 40x^3$. If we differentiate $f'(x) = 40x^3$, we obtain $f''(x) = 120x^2$. This new function is called the second derivative of $f(x) = 10x^4$.

For $y = 2x^3 - 5x^2$, the first derivative is $\frac{dy}{dx} = 6x^2 - 10x$, and the second derivative is $\frac{d^2y}{dx^2} = 12x - 10$.

Note the appearance of the superscripts in the second derivative. The reason for this choice of notation is that the second derivative is the derivative of the first derivative; that is, we write $\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$.

Other notations used to represent first and second derivatives of $y = f(x)$ are $\frac{dy}{dx} = f'(x) = y'$ and $\frac{d^2y}{dx^2} = f''(x) = y''$.

EXAMPLE 1 Selecting a strategy to determine the second derivative of a rational function

Determine the second derivative of $f(x) = \frac{x}{1 + x}$ when $x = 1$.

Solution
Write $f(x) = \frac{x}{1 + x}$ as a product and differentiate.

\[
\begin{align*}
  f(x) &= x(x + 1)^{-1} \\
  f'(x) &= (1)(x + 1)^{-1} + (x)(-1)(x + 1)^{-2}(1) \\
  &= \frac{1}{x + 1} - \frac{x}{(x + 1)^2}
\end{align*}
\]

(Product and power of a function rule)
Differentiating again to determine the second derivative,

\[ f''(x) = -2(1 + x)^{-3}(1) \]  
\[ = -2 \quad (1 + x)^{-3} \]  
\[ = \frac{-2}{(1 + x)^3} \]  
\[ \text{Evaluate} \]

When \( x = 1 \), \( f''(1) = \frac{-2}{(1 + 1)^3} \)
\[ = \frac{-2}{8} \]
\[ = \frac{-1}{4} \]

**Velocity and Acceleration—Motion on a Straight Line**

One reason for introducing the derivative is the need to calculate rates of change. Consider the motion of an object along a straight line. Examples are a car moving along a straight section of road, a ball dropped from the top of a building, and a rocket in the early stages of flight.

When studying motion along a line, we assume the object is moving along a coordinate line, which gives us an origin of reference, and positive and negative directions. The position of the object on the line relative to the origin is a function of time, \( t \), and is commonly denoted by \( s(t) \).

The rate of change of \( s(t) \) with respect to time is the object’s **velocity**, \( v(t) \), and the rate of change of the velocity with respect to time is its **acceleration**, \( a(t) \). The absolute value of the velocity is called **speed**.
Motion on a Straight Line

An object that moves along a straight line with its position determined by a function of time \( s(t) \), has a velocity of \( v(t) = s'(t) \) and an acceleration of \( a(t) = v'(t) = s''(t) \).

In Leibniz notation,

\[
\begin{align*}
v &= \frac{ds}{dt} \\
a &= \frac{dv}{dt} = \frac{d^2s}{dt^2}.
\end{align*}
\]

The speed of an object is \( |v(t)| \).

The units of velocity are displacement divided by time; typical units are m/s.

The units of acceleration are displacement divided by (time)²; typical units are metre per second per second, or metre per second squared, or m/s².

Since we are assuming that the motion is along the coordinate line, it follows that, if \( v(t) > 0 \), the object is moving to the right, and if \( v(t) < 0 \), the object is moving to the left. If \( v(t) = 0 \), the object is stationary, or at rest.

The object is accelerating when \( a(t) \) and \( v(t) \) are both positive or both negative. That is, the product of \( a(t) \) and \( v(t) \) is positive.

The object is decelerating when \( a(t) \) is positive and \( v(t) \) is negative, or when \( a(t) \) is negative and \( v(t) \) is positive. The product of \( a(t) \) and \( v(t) \) is negative.

EXAMPLE 2

Reasoning about the motion of an object along a straight line

An object is moving along a straight line. Its position, \( s(t) \), to the right of a fixed point is given by the graph shown.

When is the object moving to the right, when is it moving to the left, and when is it at rest?

Solution

The object is moving to the right whenever \( s(t) \) is increasing, or \( v(t) > 0 \).

From the graph, \( s(t) \) is increasing for \( 0 < t < 2 \) and for \( t > 6 \).

For \( 2 < t < 6 \), the value of \( s(t) \) is decreasing, or \( v(t) < 0 \), so the object is moving to the left.

At \( t = 2 \), the direction of motion of the object changes from right to left, \( v(t) = 0 \), so the object is stationary at \( t = 2 \).

At \( t = 6 \), the direction of motion of the object changes from left to right, \( v(t) = 0 \), so the object is stationary at \( t = 6 \).
EXAMPLE 3  \textbf{Connecting motion to displacement, velocity, and acceleration}\n
The position of an object moving on a line is given by \( s(t) = 6t^2 - t^3, \) \( t \geq 0, \) where \( s \) is in metres and \( t \) is in seconds.

\begin{itemize}
  \item[a.] Determine the object’s velocity and acceleration at \( t = 2. \)
  \item[b.] At what time(s) is the object at rest?
  \item[c.] In which direction is the object moving at \( t = 5? \)
  \item[d.] When is the object moving in a positive direction?
  \item[e.] When does the object return to its initial position?
\end{itemize}

\textbf{Solution}

\begin{itemize}
  \item[a.] The velocity at time \( t \) is \( v(t) = s'(t) = 12t - 3t^2. \)

  \hspace{1cm} At \( t = 2, \) \( v(2) = 12(2) - 3(2)^2 = 12. \)

  \hspace{1cm} The acceleration at time \( t \) is \( a(t) = v'(t) = s''(t) = 12 - 6t. \)

  \hspace{1cm} At \( t = 2, \) \( a(2) = 12 - 6(2) = 0. \)

  \hspace{1cm} At \( t = 2, \) the velocity is 12 m/s and the acceleration is 0 m/s\(^2\).

  \hspace{1cm} We note that, at \( t = 2, \) the object is moving at a constant velocity, neither speeding up nor slowing down.

  \item[b.] The object is at rest when the velocity is 0, that is, \( v(t) = 0. \)

  \hspace{1cm} \( 12t - 3t^2 = 0 \)

  \hspace{1cm} \( 3t(4 - t) = 0 \)

  \hspace{1cm} \( t = 0 \) or \( t = 4 \)

  \hspace{1cm} The object is at rest at \( t = 0 \) s and at \( t = 4 \) s.

  \item[c.] To determine the direction of motion, we use the velocity at time \( t = 5. \)

  \hspace{1cm} \( v(5) = 12(5) - 3(5)^2 \)

  \hspace{1cm} \( = -15 \)

  \hspace{1cm} The object is moving in a negative direction at \( t = 5. \)
\end{itemize}
d. The object moves in a positive direction when \( v(t) > 0 \); that is, when
\[
12t - 3t^2 > 0. \quad \text{(Divide by \(-3\))}
\]
\[
t^2 - 4t < 0 \quad \text{(Factor)}
\]
\[
t(t - 4) < 0
\]

There are two cases to consider since a product is negative when the first factor is positive and the second is negative and vice versa.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t &gt; 0 ) and ( t - 4 &lt; 0 )</td>
<td>( t &lt; 0 ) and ( t - 4 &gt; 0 )</td>
</tr>
<tr>
<td>so ( t &gt; 0 ) and ( t &lt; 4 )</td>
<td>so ( t &lt; 0 ) and ( t &gt; 4 )</td>
</tr>
<tr>
<td>( 0 &lt; t &lt; 4 )</td>
<td>no solution</td>
</tr>
</tbody>
</table>

Therefore, \( 0 < t < 4 \).

The graph of the velocity function is a parabola opening downward, as shown.

![Graph of velocity function with vertex at (3, 12)](image)

From the graph, we conclude that \( v(t) > 0 \) for \( 0 < t < 4 \).

The object is moving to the right during the interval \( 0 < t < 4 \).

e. At \( t = 0 \), \( s(0) = 0 \). Therefore, the object’s initial position is at 0.

To find other times when the object is at this point, we solve \( s(t) = 0 \).
\[
6t^2 - t^3 = 0 \quad \text{(Factor)}
\]
\[
t^2(6 - t) = 0 \quad \text{(Solve)}
\]
\[
t = 0 \text{ or } t = 6
\]

The object returns to its initial position after 6 s.

**EXAMPLE 4**  
**Analyzing motion along a horizontal line**

Discuss the motion of an object moving on a horizontal line if its position is given by \( s(t) = t^2 - 10t \), \( 0 \leq t \leq 12 \), where \( s \) is in metres and \( t \) is in seconds. Include the initial velocity, final velocity, and any acceleration in your discussion.
Solution
The initial position of the object occurs at time $t = 0$. Since $s(0) = 0$, the object starts at the origin.

The velocity at time $t$ is $v(t) = s'(t) = 2t - 10 = 2(t - 5)$.

The object is at rest when $v(t) = 0$.

$2(t - 5) = 0$

$t = 5$

$v(t) > 0$ for $5 < t \leq 12$, therefore the object is moving to the right.

$v(t) < 0$ for $0 \leq t < 5$, therefore the object is moving to the left.

The initial velocity is $v(0) = -10$.

At $t = 12$, $v(12) = 2(12) - 10 = 14$.

So the final velocity is 14 m/s. The velocity graph is shown below.

The acceleration at time $t$ is $a(t) = v'(t) = s''(t) = 2$.

The object moves to the left for $0 \leq t < 5$ and to the right for $5 < t \leq 12$. The initial velocity is $-10$ m/s, the final velocity is 14 m/s, and the acceleration is 2 m/s$^2$.

To draw a diagram of the motion, determine the object’s position at $t = 5$ and $t = 12$ s. (The actual path of the object is back and forth on a line.)
EXAMPLE 5  Analyzing motion under gravity near the surface of the earth

A baseball is hit vertically upward. The position function in metres, of the ball is \( s(t) = -5t^2 + 30t + 1 \) where \( t \) is in seconds.

a. Determine the maximum height reached by the ball.
b. Determine the velocity of the ball when it is caught 1 m above the ground.

Solution

a. The maximum height occurs when the velocity of the ball is zero, that is, when the slope of the tangent to the graph is zero.

The velocity function is \( v(t) = s'(t) = -10t + 30 \).

On solving \( v(t) = 0 \), we obtain \( t = 3 \).

\[
s(3) = -5(3)^2 + 30(3) + 1 = 46
\]

Therefore, the maximum height reached by the ball is 46 m.

b. When the ball is caught, \( s(t) = 1 \). To find the time at which this occurs, solve

\[
1 = -5t^2 + 30t + 1 \\
0 = -5t(t - 6) \\
t = 0 \text{ or } t = 6.
\]

Since \( t = 0 \) is the time at which the ball leaves the bat, the time at which the ball is caught is \( t = 6 \).

The velocity of the ball when it is caught is \( v(6) = -10(6) + 30 = -30 \) m/s.

This negative value is reasonable, since the ball is falling (moving in a negative direction) when it is caught.

Note, however, that the graph of \( s(t) \) does not represent the path of the ball. We think of the ball as moving in a straight line along a vertical \( s \)-axis, with the direction of motion reversing when \( s = 46 \).

To see this, note that the ball is at the same height at time \( t = 1 \), when \( s(1) = 26 \), and at time \( t = 5 \), when \( s(5) = 26 \).
IN SUMMARY

Key Ideas

- The derivative of the derivative function is called the second derivative.
- If the position of an object, s(t), is a function of time t, then the first derivative of this function represents the velocity of the object at time t: \( v(t) = s'(t) = \frac{ds}{dt} \).
- Acceleration, a(t), is the instantaneous rate of change of velocity with respect to time. Acceleration is the first derivative of the velocity function and the second derivative of the position function:
  \[ a(t) = v'(t) = s''(t), \text{ or } a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \]

Need to Know

- Negative velocity, \( v(t) < 0 \) or \( s'(t) < 0 \), indicates that an object is moving in a negative direction (left or down).
- Positive velocity, \( v(t) > 0 \) or \( s'(t) > 0 \), indicates that an object is moving in a positive direction (right or up).
- Zero velocity, \( v(t) = 0 \) or \( s'(t) = 0 \), indicates that an object is stationary and that a possible change in direction may occur.
- Notations for the second derivative are \( f''(x), \frac{d^2y}{dx^2}, \frac{d^2}{dx^2}[f(x)], \) or \( y'' \).
- Negative acceleration, \( a(t) < 0 \) or \( v'(t) < 0 \), indicates that the velocity is decreasing.
- Positive acceleration, \( a(t) > 0 \) or \( v'(t) > 0 \), indicates that the velocity is increasing.
- Zero acceleration, \( a(t) = 0 \) or \( v'(t) = 0 \), indicates that the velocity is constant and the object is neither accelerating nor decelerating.
- An object is accelerating (speeding up) when its velocity and acceleration have the same signs.
- An object is decelerating (slowing down) when its velocity and acceleration have opposite signs.
Exercise 3.1

PART A

1. Explain and discuss the difference in velocity at times $t = 1$ and $t = 5$ for $v(t) = 2t - t^2$.

2. Determine the second derivative of each of the following:
   a. $y = x^{10} + 3x^6$
   b. $f(x) = \sqrt{x}$
   c. $y = (1 - x)^2$
   d. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$
   e. $y = 4x^3 - x^{-2}$
   f. $f(x) = \frac{2x}{x + 1}$
   g. $y = x^2 + \frac{1}{x^2}$
   h. $g(x) = \sqrt{3x - 6}$
   i. $y = (2x + 4)^3$
   j. $h(x) = \sqrt[3]{x^5}$

3. For the following position functions, each of which describes the motion of an object along a straight line, find the velocity and acceleration as functions of $t$, $t \geq 0$.
   a. $s(t) = 5t^2 - 3t + 15$
   b. $s(t) = 2t^3 + 36t - 10$
   c. $s(t) = t - 8 + \frac{6}{t}$
   d. $s(t) = (t - 3)^2$
   e. $s(t) = \sqrt{t} + 1$
   f. $s(t) = \frac{9t}{t + 3}$

4. Consider the following position versus time graphs.

   a. 
   
   b. 

   i. When is the velocity zero?
   ii. When is the object moving in a positive direction?
   iii. When is the object moving in a negative direction?
5. A particle moves along a straight line with the equation of motion
\[ s = \frac{1}{3}t^3 - 2t^2 + 3t, \quad t \geq 0. \]
   a. Determine the particle’s velocity and acceleration at any time \( t \).
   b. When does the motion of the particle change direction?
   c. When does the particle return to its initial position?

**PART B**

6. Each function describes the position of an object that moves along a straight line. Determine whether the object is moving in a positive or negative direction at time \( t = 1 \) and at time \( t = 4 \).
   a. \( s(t) = -\frac{1}{3}t^2 + t + 4 \)
   b. \( s(t) = t(t - 3)^2 \)
   c. \( s(t) = t^3 - 7t^2 + 10t \)

7. Starting at \( t = 0 \), a particle moves along a line so that its position after \( t \) seconds is \( s(t) = t^2 - 6t + 8 \), where \( s \) is in metres.
   a. What is its velocity at time \( t \)?
   b. When is its velocity zero?

8. When an object is launched vertically from ground level with an initial velocity of 40 m/s, its position after \( t \) seconds will be \( s(t) = 40t - 5t^2 \) metres above ground level.
   a. When does the object stop rising?
   b. What is its maximum height?

9. An object moves in a straight line, and its position, \( s \), in metres after \( t \) seconds is \( s(t) = 8 - 7t + t^2 \).
   a. Determine the velocity when \( t = 5 \)
   b. Determine the acceleration when \( t = 5 \).

10. The position function of a moving object is \( s(t) = t^2(7 - t), \quad t \geq 0 \), in metres, at time \( t \) in seconds.
    a. Calculate the object’s velocity and acceleration at any time \( t \).
    b. After how many seconds does the object stop?
    c. When does the motion of the object change direction?
    d. When is its acceleration positive?
    e. When does the object return to its original position?

11. A ball is thrown upward, and its height, \( h \), in metres above the ground after \( t \) seconds is given by \( h(t) = -5t^2 + 25t, \quad t \geq 0 \).
    a. Calculate the ball’s initial velocity.
    b. Calculate its maximum height.
    c. When does the ball strike the ground, and what is its velocity at this time?
12. A dragster races down a 400 m strip in 8 s. Its distance, in metres, from the starting line after \( t \) seconds is \( s(t) = 6t^2 + 2t \).

a. Determine the dragster’s velocity and acceleration as it crosses the finish line.

b. How fast was it moving 60 m down the strip?

13. For each of the following position functions, discuss the motion of an object moving on a horizontal line, where \( s \) is in metres and \( t \) is in seconds. Make a graph similar to that in Example 4, showing the motion for \( t \geq 0 \). Find the velocity and acceleration, and determine the extreme positions (farthest left or right) for \( t \geq 0 \).

a. \( s(t) = 10 + 6t - t^2 \)  
b. \( s(t) = t^3 - 12t - 9 \)

14. If the position function of an object is \( s(t) = t^5 - 10t^2 \), at what time, \( t \), in seconds, will the acceleration be zero? Is the object moving toward or away from the origin at that instant?

15. The distance–time relationship for a moving object is given by \( s(t) = kt^2 + (6k^2 - 10k)t + 2k \), where \( k \) is a non-zero constant.

a. Show that the acceleration is constant.

b. Find the time at which the velocity is zero, and determine the position of the object when this occurs.

PART C

16. An elevator is designed to start from a resting position without a jerk. It can do this if the acceleration function is continuous.

a. Show that, for the position function \( s(t) = \begin{cases} 0, & \text{if } t < 0 \\ \frac{t^3}{t^2 + 1}, & \text{if } t \geq 0 \end{cases} \),
the acceleration is continuous at \( t = 0 \).

b. What happens to the velocity and acceleration for very large values of \( t \)?

17. An object moves so that its velocity, \( v \), is related to its position, \( s \), according to \( v = \sqrt{b^2 + 2gs} \), where \( b \) and \( g \) are constants. Show that the acceleration of the object is constant.

18. Newton’s law of motion for a particle of mass \( m \) moving in a straight line says that \( F = ma \), where \( F \) is the force acting on the particle and \( a \) is the acceleration of the particle. In relativistic mechanics, this law is replaced by

\[
F = \frac{m_0 \frac{dv}{dt}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},
\]

where \( m_0 \) is the mass of the particle measured at rest and \( c \) is the velocity of light. Show that

\[
F = \frac{m_0a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}},
\]
Section 3.2—Maximum and Minimum on an Interval (Extreme Values)

INVESTIGATION

The purpose of this investigation is to determine how the derivative can be used in determining the maximum (largest) value or the minimum (smallest) value of a function on a given interval. Together, these are called the absolute extrema on an interval.

A. For each of the following functions, determine, by completing the square, the value of $x$ that produces a maximum or minimum function value on the given interval.
   a. $f(x) = -x^2 + 6x - 3$, $0 \leq x \leq 5$
   b. $f(x) = -x^2 - 2x + 11$, $-3 \leq x \leq 4$
   c. $f(x) = 4x^2 - 12x + 7$, $-1 \leq x \leq 4$

B. For each function in part A, determine the value of $c$ such that $f'(c) = 0$.

C. Compare the values obtained in parts A and B for each function. Why does it make sense to say that the pattern you discovered is not merely a coincidence?

D. Using a graphing calculator, graph each of the following functions and determine all values of $x$ that produce a maximum or minimum function value on the given interval.
   a. $f(x) = x^3 - 3x^2 - 8x + 10$, $-2 \leq x \leq 4$
   b. $f(x) = x^3 - 12x + 5$, $-3 \leq x \leq 3$
   c. $f(x) = 3x^3 - 15x^2 + 9x + 23$, $0 \leq x \leq 4$
   d. $f(x) = -2x^3 + 12x + 7$, $-2 \leq x \leq 2$
   e. $f(x) = -x^3 - 2x^2 + 15x + 23$, $-4 \leq x \leq 3$

E. For each function in part D, determine all values of $c$ such that $f'(c) = 0$.

F. Compare the values obtained in parts D and E for each function.

G. From your conclusions in parts C and F, state a method for using the derivative of a function to determine values of the variable that determine maximum or minimum values of the function.
H. Repeat part D for the following functions, using the indicated intervals.

a. \( f(x) = -x^2 + 6x - 3, \ 4 \leq x \leq 8 \)

b. \( f(x) = 4x^2 - 12x + 7, \ 2 \leq x \leq 6 \)

c. \( f(x) = x^3 - 3x^2 - 9x + 10, \ -2 \leq x \leq 6 \)

d. \( f(x) = x^3 - 12x + 5, \ 0 \leq x \leq 5 \)

e. \( f(x) = x^3 - 5x^2 + 3x + 7, \ -2 \leq x \leq 5 \)

I. In parts C and F, you saw that a maximum or minimum can occur at points \((c, f(c))\) where \(f'(c) = 0\). From your observations in part H, state other values of the variable that can produce a maximum or minimum in a given interval.

Check Your Understanding

The maximum value of a function that has a derivative at all points in an interval occurs at a “peak” \((f'(c) = 0)\) or at an end point of the interval. The minimum value occurs at a “valley” \((f'(c) = 0)\) or at an end point. This is true no matter how many peaks and valleys the graph has in the interval.

In the following three graphs, the derivative equals zero at two points.

![Graphs with peaks and valleys](image)

Algorithm for Finding Maximum or Minimum (Extreme) Values

If a function \(f(x)\) has a derivative at every point in the interval \(a \leq x \leq b\), calculate \(f(x)\) at

- all points in the interval \(a \leq x \leq b\) where \(f'(x) = 0\)
- the end points \(x = a\) and \(x = b\)

The maximum value of \(f(x)\) on the interval \(a \leq x \leq b\) is the largest of these values, and the minimum value of \(f(x)\) on the interval is the smallest of these values.
EXAMPLE 1  Selecting a strategy to determine absolute extrema

Find the extreme values of the function \( f(x) = -2x^3 + 9x^2 + 4 \) on the interval \([1, 5]\).

Solution

The derivative is \( f'(x) = -6x^2 + 18x \).

If we set \( f'(x) = 0 \), we obtain \( -6x(x - 3) = 0 \), so \( x = 0 \) or \( x = 3 \).

Both values lie in the given interval, \([1, 5]\).

We can then evaluate \( f(x) \) for these values and at the end points \( x = -1 \) and \( x = 5 \) to obtain

\[ f(-1) = 15 \]
\[ f(0) = 4 \]
\[ f(3) = 31 \]
\[ f(5) = -21 \]

Therefore, the maximum value of \( f(x) \) on the interval \(-1 \leq x \leq 5\) is \( f(3) = 31 \), and the minimum value is \( f(5) = -21 \).

Graphing the function on this interval verifies our analysis.

EXAMPLE 2  Solving a problem involving absolute extrema

The amount of current (in amperes) in an electrical system is given by the function \( C(t) = -t^3 + t^2 + 21t \), where \( t \) is the time in seconds and \( 0 \leq t \leq 5 \).

Determine the times at which the current is at its maximum and minimum, and determine the amount of current in the system at these times.

Solution

The derivative is \( \frac{dC}{dt} = -3t^2 + 2t + 21 \).

If we set \( \frac{dC}{dt} = 0 \), we obtain

\[ -3t^2 + 2t + 21 = 0 \quad \text{(Multiply by \(-1\))} \]
\[ 3t^2 - 2t - 21 = 0 \quad \text{(Factor)} \]
\[ (3t + 7)(t - 3) = 0 \quad \text{(Solve)} \]

therefore, \( t = -\frac{7}{3} \) or \( t = 3 \).
Only \( t = 3 \) is in the given interval, so we evaluate \( C(t) \) at \( t = 0, t = 3, \) and \( t = 5 \) as follows:

\[
C(0) = 0 \\
C(3) = -3^3 + 3^2 + 21(3) = 45 \\
C(5) = -5^3 + 5^2 + 21(5) = 5
\]

The maximum is 45 amperes at time \( t = 3 \) s, and the minimum is zero amperes at time \( t = 0 \) s.

Graphing the function on this interval verifies our analysis.

---

**EXAMPLE 3**  
Selecting a strategy to determine the absolute minimum

The amount of light intensity on a point is given by the function

\[ I(t) = \frac{t^2 + 2t + 16}{t + 2}, \]

where \( t \) is the time in seconds and \( t \in [0, 14] \). Determine the time of minimal intensity.

**Solution**

Note that the function is not defined for \( t = -2 \). Since this value is not in the given interval, we need not worry about it.

The derivative is

\[
I'(t) = \frac{(2t + 2)(t + 2) - (t^2 + 2t + 16)(1)}{(t + 2)^2} \\
= \frac{2t^2 + 6t + 4 - t^2 - 2t - 16}{(t + 2)^2} \\
= \frac{t^2 + 4t - 12}{(t + 2)^2}
\]

If we set \( I'(t) = 0 \), we only need to consider when the numerator is 0 and obtain

\[
t^2 + 4t - 12 = 0 \quad \text{(Factor)} \\
(t + 6)(t - 2) = 0 \quad \text{(Solve)} \\
t = -6 \text{ or } t = 2
\]
Only \( t = 2 \) is in the given interval, so we evaluate \( I(t) \) for \( t = 0, 2, \) and 14.

\[
I(0) = 8 \\
I(2) = \frac{4 + 4 + 16}{4} = 6 \\
I(14) = \frac{14^2 + 2(14) + 16}{16} = 15
\]

Note that the calculation can be greatly reduced by rewriting the intensity function, as shown.

\[
I(t) = \frac{t^2 + 2t + 16}{t+2} + \frac{16}{t+2} = t + 16(t+2)^{-1}
\]

Then \( I'(t) = 1 - 16(t+2)^{-2} \)

\[
= 1 - \frac{16}{(t+2)^2}
\]

Setting \( I'(t) = 0 \) gives

\[
1 = \frac{16}{(t+2)^2}
\]

\[
t^2 + 4t + 4 = 16 \\
t^2 + 4t - 12 = 0
\]

As before, \( t = -6 \) or \( t = 2 \).

The evaluations are also simplified:

\[
I(0) = 0 + \frac{16}{2} = 8 \\
I(2) = 2 + \frac{16}{4} = 6 \\
I(14) = 14 + \frac{16}{16} = 15
\]

Either way, the minimum amount of light intensity occurs at \( t = 2 \) s on the given time interval.
IN SUMMARY

Key Ideas

- The **maximum and minimum values** of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a “peak” \( f'(c) = 0 \) or at an end point of the interval.
- The minimum value occurs at a “valley” \( f'(c) = 0 \) or at an end point of the interval.

Need to Know

- **Algorithm for Finding Extreme Values**
  For a function \( f(x) \) that has a derivative at every point in the interval, the maximum or minimum values can be found by using the following procedure.
  1. **Determine** \( f'(x) \). Find all points in the interval \( a \leq x \leq b \) where \( f'(x) = 0 \).
  2. **Evaluate** \( f(x) \) at the endpoints \( a \) and \( b \), and at points where \( f'(x) = 0 \).
  3. **Compare** all the values found in step 2.
     - The largest value is the maximum value of \( f(x) \) on the interval \( a \leq x \leq b \).
     - The smallest value is the minimum value of \( f(x) \) on the interval \( a \leq x \leq b \).

Exercise 3.2

**PART A**

1. State, with reasons, why the maximum/minimum algorithm can or cannot be used to determine the maximum and minimum values for the following functions.
   a. \( y = x^3 - 5x^2 + 10, -5 \leq x \leq 5 \)
   b. \( y = \frac{3x}{x - 2}, -1 \leq x \leq 3 \)
   c. \( y = \frac{x}{x^2 - 4}, x \in [0, 5] \)
   d. \( y = \frac{x^2 - 1}{x + 3}, x \in [-2, 3] \)
2. State the absolute maximum value and the absolute minimum value for each function. In each of the following graphs, the function is defined in the interval shown.

![Graph a](image)

![Graph c](image)

![Graph b](image)

![Graph d](image)

3. Determine the absolute extrema of each function on the given interval. Illustrate your results by sketching the graph of each function.

   a. \( f(x) = x^2 - 4x + 3, \ 0 \leq x \leq 3 \)
   b. \( f(x) = (x - 2)^2, \ 0 \leq x \leq 2 \)
   c. \( f(x) = x^3 - 3x^2, \ -1 \leq x \leq 3 \)
   d. \( f(x) = x^3 - 3x^2, \ x \in [-2, 1] \)
   e. \( f(x) = 2x^3 - 3x^2 - 12x + 1, \ x \in [-2, 0] \)
   f. \( f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, \ x \in [0, 4] \)
PART B

4. Using the algorithm for finding maximum or minimum values, determine the absolute extreme values of each function on the given interval.
   a. $f(x) = x + \frac{4}{x}, \; 1 \leq x \leq 10$
   b. $f(x) = 4\sqrt{x} - x, \; x \in [2, 9]$
   c. $f(x) = \frac{1}{x^2 - 2x + 2}, \; 0 \leq x \leq 2$
   d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20, \; x \in [-3, 4]$
   e. $f(x) = \frac{4x}{x^2 + 1}, \; -2 \leq x \leq 4$
   f. $f(x) = \frac{4x}{x^2 + 1}, \; x \in [2, 4]$

5. a. An object moves in a straight line. Its velocity in m/s at time $t$ is $v(t) = \frac{4t^2}{4 + t^2}, \; t \geq 0$. Determine the maximum and minimum velocities over the time interval $1 \leq t \leq 4$.
   b. Repeat 5. a. where $v(t) = \frac{4t^2}{1 + t^2}, \; t \geq 0$.

6. A swimming pool is treated periodically to control the growth of bacteria. Suppose that $t$ days after a treatment, the concentration of bacteria per cubic centimetre is $C(t) = 30t^2 - 240t + 500$. Determine the lowest concentration of bacteria during the first week after the treatment.

7. The fuel efficiency, $E$, (in litres per 100 kilometres) of a car driven at speed $v$ (in km/h) is $E(v) = \frac{1600v}{v^2 + 6400}$.
   a. If the speed limit is 100 km/h, determine the legal speed that will maximize the fuel efficiency.
   b. Repeat 7. a. using a speed limit of 50 km/h.
   c. Determine the speed intervals within the legal speed limit in which the fuel efficiency $E$ is increasing.
   d. Determine the speed intervals within the legal speed limit in which the fuel efficiency $E$ is decreasing.

8. The concentration $C(t)$ (in milligrams per cubic centimetre) of a certain medicine in a patient’s bloodstream is given by $C(t) = \frac{0.1t}{(t + 3)^2}$, where $t$ is the number of hours after the medicine is taken. Determine the maximum and minimum concentrations between the first and sixth hours after the patient is given the medicine.
9. Technicians working for the Ministry of Natural Resources found that the amount of a pollutant in a certain river can be represented by

\[ P(t) = 2t + \frac{1}{(162t + 1)}, \quad 0 \leq t \leq 1, \] where \( t \) is the time (in years) since a clean-up campaign started. At what time was the pollution at its lowest level?

10. A truck travelling at \( x \) km/h, where \( 30 \leq x \leq 120 \), uses gasoline at the rate of \( r(x) \) L/100 km where \( r(x) = \frac{4900}{x} + x \). If fuel costs $1.15/L, what speed will result in the lowest fuel cost for a trip of 200 km? What is the lowest total cost for the trip?

11. The polynomial function \( f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10, \quad 0 \leq x \leq 75 \), models the shape of a roller coaster track, where \( f \) is the vertical displacement of the track and \( x \) is the horizontal displacement of the track. Both displacements are in metres. Determine the absolute maximum and minimum heights along this stretch of track.

12. a. Graph the cubic function with an absolute minimum at \((-2, -12)\), a local maximum at \((0, 3)\), a local minimum at \((2, -1)\), and an absolute maximum at \((4, 9)\).

b. What is the domain of this function?

c. Where is the function increasing? decreasing?

13. What points on an interval must you consider to determine the absolute maximum or minimum value on the interval? Why?

**PART C**

14. In a certain manufacturing process, when the level of production is \( x \) units, the cost of production (in dollars) is \( C(x) = 3000 + 9x + 0.05x^2, \quad 1 \leq x \leq 300 \).

What level of production, \( x \), will minimize the unit cost \( U(x) = \frac{C(x)}{x} \)? Keep in mind that the production level must be an integer.

15. Repeat question 14. where the cost of production is \( C(x) = 6000 + 9x + 0.05x^2, \quad 1 \leq x \leq 300 \).
1. Determine the second derivative of each of the following functions.
   a. \( h(x) = 3x^4 - 4x^3 - 3x^2 - 5 \)
   b. \( f(x) = (2x - 5)^3 \)
   c. \( y = \frac{15}{x + 3} \)
   d. \( g(x) = \sqrt{x^2 + 1} \)

2. The displacement of an object in motion is described by \( s(t) = t^3 - 21t^2 + 90t \), where the displacement, \( s \), is measured in metres at \( t \) seconds.
   a. Calculate the displacement at 3 s.
   b. Calculate the velocity at 5 s.
   c. Calculate the acceleration at 4 s.

3. A ball is thrown upward. Its motion can be described by \( h(t) = -4.9t^2 + 6t + 2 \), where the height, \( h \), is measured in metres at \( t \) seconds.
   a. Determine the initial velocity.
   b. When does the ball reach its maximum height?
   c. When does the ball hit the ground?
   d. What is its velocity when it hits the ground?
   e. What is the acceleration of the ball on the way up? on the way down?

4. An object is moving horizontally. The object’s displacement, \( s \), in metres at \( t \) seconds is described by \( s(t) = 4t - 7t^2 + 2t^3 \).
   a. Determine the velocity and acceleration at \( t = 2 \).
   b. When is the object stationary? Describe the motion immediately before and after these times.
   c. At what time, to the nearest tenth of a second, is the acceleration equal to 0? Describe the motion at that time.

5. Determine the absolute extreme values of each function on the given interval, using the algorithm for finding maximum and minimum values.
   a. \( f(x) = x^3 + 3x^2 + 1, -2 \leq x \leq 2 \)
   b. \( f(x) = (x + 2)^2, -3 \leq x \leq 3 \)
   c. \( f(x) = \frac{1}{x} - \frac{1}{x^3}, x \in [1, 5] \)

6. The volume, \( V \), of 1 kg of H\(_2\)O at temperature \( t \) between 0 °C and 30 °C can be modelled by \( V(t) = -0.000\ 067t^3 + 0.008\ 504\ 3t^2 - 0.064\ 26t + 999.87 \). Volume is measured in cubic centimetres. Determine the temperature at which the volume of water is the greatest in the given interval.
7. For each function, evaluate
   a. \( f'(3) \) if \( f(x) = x^4 - 3x \)
   b. \( f'(-2) \) if \( f(x) = 2x^3 + 4x^2 - 5x + 8 \)
   c. \( f''(1) \) if \( f(x) = -3x^2 - 5x + 7 \)
   d. \( f''(-3) \) if \( f(x) = 4x^3 - 3x^2 + 2x - 6 \)
   e. \( f'(0) \) if \( f(x) = 14x^2 + 3x - 6 \)
   f. \( f''(4) \) if \( f(x) = x^4 + x^5 - x^3 \)
   g. \( f''\left(\frac{1}{3}\right) \) if \( f(x) = -2x^5 + 2x - 6 - 3x^3 \)
   h. \( f'\left(\frac{3}{4}\right) \) if \( f(x) = -3x^3 - 7x^2 + 4x - 11 \)

8. On the surface of the moon, an astronaut can jump higher because the force of gravity is less than it is on Earth. When an astronaut jumps, his or her height in metres above the moon’s surface can be modelled by
   \[ s(t) = t\left(-\frac{5}{6}t + 1\right), \] where \( t \) is measured in seconds. What is the acceleration due to gravity on the moon?

9. The forward motion of a space shuttle \( t \) seconds after touchdown is described by \( s(t) = 189t - t^3 \), where \( s \) is measured in metres.
   a. What is the velocity of the shuttle at touchdown?
   b. How much time is required for the shuttle to stop completely?
   c. How far does it travel from touchdown to a complete stop?
   d. What is the deceleration eight seconds after touchdown?

10. The team skip slides a curling stone toward the rings at the opposite end of the ice. The stone’s position, \( s \), in metres at \( t \) seconds can be modelled by \( s(t) = 12t - 4t^2 \). How far does the stone travel before it stops? For how long is it moving?

11. After a football is punted, its height, \( h \), in metres above the ground at \( t \) seconds can be modelled by \( h(t) = -4.9t^2 + 21t + 0.45 \).
   a. Determine the restricted domain of this model.
   b. When does the ball reach its maximum height?
   c. What is the ball’s maximum height?
Section 3.3—Optimization Problems

We frequently encounter situations in which we are asked to do the best we can. Such a request is vague unless we are given some conditions. Asking us to minimize the cost of making tables and chairs is not clear. Asking us to make the maximum number of tables and chairs possible so that the costs of production are minimized and given that the amount of material available is restricted allows us to construct a function describing the situation. We can then determine the minimum (or maximum) of the function.

Such a procedure is called optimization. To optimize a situation is to realize the best possible outcome, subject to a set of restrictions. Because of these restrictions, the domain of the function is usually restricted. As you have seen earlier, in such situations, the maximum or minimum can be identified through the use of calculus, but might also occur at the ends of the restricted domain.

EXAMPLE 1 Solving a problem involving optimal area

A farmer has 800 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road that is already fenced, so the farmer needs to fence only the remaining three sides of the field. The farmer wants to enclose the maximum possible area and to use all the fencing. How does the farmer determine the dimensions that achieve this?

Solution

The farmer can achieve the goal by determining a function that describes the area, subject to the condition that the amount of fencing used is to be exactly 800 m, and by finding the maximum of the function. To do so, the farmer proceeds as follows:

Let the width of the enclosed area be \( x \) metres.

Then the length of the rectangular field is \( (800 - 2x) \) m. The area of the field can be represented by the function \( A(x) \) where

\[
A(x) = x(800 - 2x) = 800x - 2x^2
\]

The domain of the function is \( 0 < x < 400 \), since the amount of fencing is 800 m. To find the minimum and maximum values, determine \( A'(x) \): \( A'(x) = 800 - 4x \). Setting \( A'(x) = 0 \), we obtain \( 800 - 4x = 0 \), so \( x = 200 \).
The minimum and maximum values can occur at \( x = 200 \) or at the ends of the domain, \( x = 0 \) and \( x = 400 \). Evaluating the area function at each of these gives:

\[
A(0) = 0
\]
\[
A(200) = 200(800 - 400) = 80000
\]
\[
A(400) = 400(800 - 800) = 0
\]

Sometimes, the ends of the domain produce results that are either not possible or unrealistic. In this case \( x = 200 \) produces the maximum. The ends of the domain do not result in possible dimensions of a rectangle.

The maximum area the farmer can enclose is 80 000 m\(^2\) within a field 200 m by 400 m.

---

**EXAMPLE 2**

**Solving a problem involving optimal volume**

A piece of sheet metal 60 cm by 30 cm is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box with the largest volume.

![Diagram of sheet metal with squares cut out](image)

**Solution**

From the diagram, making the box requires that the four corner squares be cut out and discarded. Folding up the sides creates the box. Let each side of the squares be \( x \) centimetres.

Therefore, height = \( x \)

length = \( 60 - 2x \).

width = \( 30 - 2x \).

Since all dimensions are positive, \( 0 < x < 15 \).

![Diagram of box with dimensions labeled](image)

The volume of the box is the product of its dimensions and is given by the function \( V(x) \), where

\[
V(x) = x(60 - 2x)(30 - 2x) = 4x^3 - 180x^2 + 1800x
\]
For extreme values, set \( V'(x) = 0 \).

\[
V'(x) = 12x^2 - 360x + 1800 \\
= 12(x^2 - 30x + 150)
\]

Setting \( V'(x) = 0 \), we obtain \( x^2 - 30x + 150 = 0 \). Solving for \( x \) using the quadratic formula results in

\[
x = \frac{30 \pm \sqrt{300}}{2} \\
= 15 \pm 5\sqrt{3}
\]

\( x = 23.7 \) or \( x = 6.3 \)

Since \( 0 < x < 15 \), \( x = 15 - 5\sqrt{3} \approx 6.3 \).

To find the largest volume, substitute \( x = 6.3 \) in \( V(x) = 4x^3 - 180x^2 + 1800x \).

\[
V(6.3) = 4(6.3)^3 - 180(6.3)^2 + 1800(6.3) \\
= 5196
\]

Notice that the endpoints of the domain did not have to be tested since it is impossible to make a box using the values \( x = 0 \) or \( x = 15 \).

The maximum volume is obtained by cutting out corner squares of side length 6.3 cm. The length of the box is \( 60 - 2 \times 6.3 = 47.4 \) cm, the width is \( 30 - 2 \times 6.3 = 17.4 \) cm, and the height is 6.3 cm.

---

**EXAMPLE 3**

**Solving a problem that minimizes distance**

Ian and Ada are both training for a marathon. Ian’s house is located 20 km north of Ada’s house. At 9:00 a.m. one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?

**Solution**

If Ian starts at point \( I \), he reaches point \( J \) after time \( t \) hours. Then \( IJ = 8t \) km, and \( JA = (20 - 8t) \) km.

If Ada starts at point \( A \), she reaches point \( B \) after \( t \) hours, and \( AB = 6t \) km.

Now the distance they are apart is \( s = JB \), and \( s \) can be expressed as a function of \( t \) by

\[
s(t) = \sqrt{JA^2 + AB^2} \\
= \sqrt{(20 - 8t)^2 + (6t)^2} \\
= \sqrt{100t^2 - 320t + 400 + 36t^2} \\
= \sqrt{136t^2 - 320t + 400}.
\]
The domain for \( t \) is \( 0 \leq t \leq 2.5 \).

\[
s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)
\]

\[
= \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}
\]

To obtain a minimum or maximum value, let \( s'(t) = 0 \).

\[
\frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} = 0
\]

\[
100t - 160 = 0
\]

\[
t = 1.6
\]

Using the algorithm for finding extreme values

\[
s(0) = \sqrt{400} = 20
\]

\[
s(1.6) = \sqrt{100(1.6)^2 - 320(1.6) + 400} = 12
\]

\[
s(2.5) = \sqrt{225} = 15
\]

Therefore, the minimum value of \( s(t) \) is 12 km, which occurs at time 10:36.

**IN SUMMARY**

**Key Ideas**

- In an optimization problem, you must determine the maximum or minimum value of a quantity.
- Optimization problems can be solved using mathematical models that are developed using information given in the problem. The numerical solution represents the extreme value of the model.

**Need to Know**

- *An Algorithm for Solving Optimization Problems*
  1. Understand the problem and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
  2. Whenever possible, draw a diagram, labelling the given and required quantities.
  3. Determine the domain of the function to be optimized, using the information given in the problem.
  4. Use the algorithm for extreme values to find the absolute maximum or minimum function value in the domain.
  5. Use the result of step 4 to answer the original problem.
Exercise 3.3

PART A

1. A piece of wire 100 cm long is to be bent to form a rectangle. Determine the dimensions of a rectangle with the maximum area.

2. Discuss the result of maximizing the area of a rectangle, given a fixed perimeter.

3. A farmer has 600 m of fence and wants to enclose a rectangular field beside a river. Determine the dimensions of the fenced field in which the maximum area is enclosed. (Fencing is required on only three sides.)

4. A rectangular piece of cardboard 100 cm by 40 cm is to be used to make a rectangular box with an open top by cutting congruent squares from the corners. Calculate the dimensions (to one decimal place) for a box with the largest volume.

5. A rectangle has a perimeter of 440 cm. What dimensions maximize the rectangle’s area?

6. What are the dimensions of a rectangle with an area of 64 m² and the smallest possible perimeter?

7. A rancher has 1000 m of fencing to enclose two rectangular corrals. The corrals have the same dimensions and one side in common. What dimensions will maximize the enclosed area?

8. A net enclosure for practising golf shots is open at one end, as shown. Find the dimensions that minimize the amount of netting and that give a volume of 144 m². (Netting is required only on the sides, the top, and the far end.)
PART B
9. The volume of a square-based rectangular cardboard box is to be 1000 cm³. Determine the dimensions with which the quantity of material used to manufacture all 6 faces is a minimum. Assume there will be no waste material. The machinery available cannot fabricate material smaller than 2 cm in length.

10. Determine the area of the largest rectangle that can be inscribed inside a semicircle with radius of 10 units. Place the length of the rectangle along the diameter.

11. A cylindrical-shaped tin can is to have a capacity of 1000 cm³.
   a. Determine the dimensions that require the minimum amount of tin for the can. (Assume no waste material.) The marketing department says the smallest can the market will accept has a diameter of 6 cm and a height of 4 cm.
   b. Express the answer for 11. a. as a ratio of height to diameter.

12. a. Determine the area of the largest rectangle that can be inscribed in a right triangle with legs adjacent to the right angle of lengths 5 cm and 12 cm. The two sides of the rectangle lie along the legs.
   b. Repeat 12. a. with a right triangle that has sides 8 cm and 15 cm.
   c. Hypothesize a conclusion for any right triangle.

13. a. An isosceles trapezoidal drainage gutter is to be made so that the angles at A and B in the cross-section ABCD are each 120°. If the 5 m long sheet of metal that has to be bent to form the open-topped gutter has a width of 60 cm, then determine the dimensions so the cross-sectional area will be a maximum.

   b. Calculate the maximum volume of water that can be held by this gutter.

14. A piece of window framing material is 6 m long. A carpenter wants to build a frame for a rural gothic style window where ΔABC is equilateral. The window must fit inside a space 1 m wide and 3 m high.
a. Determine the dimensions that should be used for the 6 pieces so the maximum amount of light will be admitted. Assume no waste of material for corner cuts, etc.

b. Would the carpenter get more light if he built a window in the shape of an equilateral triangle only? Explain.

15. A train leaves the station at 10:00 a.m. and travels due south at a speed of 60 km/h. Another train has been heading due west at 45 km/h and reaches the same station at 11:00 a.m. At what time were the two trains closest together?

16. A north–south highway intersects an east–west highway at point $P$. A vehicle crosses $P$ at 1:00 p.m., travelling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of $P$, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.

**PART C**

17. In question 12. c., you looked at two specific right triangles and observed that a rectangle with the maximum area that can be inscribed inside the triangle had dimensions equal to half the lengths of the sides adjacent to the rectangle. Prove this is true for any right triangle.

18. Prove that any cylindrical can of volume $k$ cubic units that is to be made using a minimum amount of material must have the height equal to the diameter.

19. A piece of wire 100 cm long is cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. Determine how the wire should be cut so the total area enclosed is

   a. a maximum
   b. a minimum

20. Determine the minimal distance from point $(-3, 3)$ to the curve given by $y = (x - 3)^2$.

21. A chord joins any two points $A$ and $B$ on the parabola whose equation is $y^2 = 4x$. If $C$ is the midpoint of $AB$, and $CD$ is drawn parallel to the $x$-axis to meet the parabola at $D$, prove that the tangent at $D$ is parallel to chord $AB$.

22. A rectangle lies in the first quadrant with one vertex at the origin and two of the sides along the coordinate axes. If the fourth vertex lies on the line defined by $x + 2y - 10 = 0$, find the rectangle with the maximum area.

23. The base of a rectangle lies along the $x$-axis, and the upper two vertices are on the curve defined by $y = k^2 - x^2$. Determine the dimensions of the rectangle with the maximum area.
In the world of business, it is extremely important to manage costs effectively. Good control will allow for minimization of costs and maximization of profit. At the same time, there are human considerations. If your company is able to maximize profit but antagonizes customers or employees in the process, there may be a significant penalty to pay in the future. For this reason, it may be important that, in addition to any mathematical constraints, you consider other more practical constraints on the domain when you construct a workable function.

The following examples will illustrate economic situations and domain constraints you may encounter.

**EXAMPLE 1** Solving a problem to maximize revenue

A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is $7.00 per person. Market research shows that 40 fewer people would ride the train for each $0.10 increase in the fare, and 40 more people would ride the train for each $0.10 decrease. If the capacity of the train is 2600 passengers, and carrying fewer than 1600 passengers means costs exceed revenue, what fare should the railway charge to get the largest possible revenue?

**Solution**

In order to maximize revenue, we require a revenue function. We know that revenue = (number of passengers) × (fare per passenger).

In forming a revenue function, the most straightforward choice for the independent variable comes from noticing that both the number of passengers and the fare per passenger change with each $0.10 increase or decrease in the fare. If we let \( x \) represent the number of $0.10 increases in the fare (e.g., \( x = 3 \) represents a $0.30 increase in the fare, whereas \( x = -1 \) represents a $0.10 decrease in the fare), then we can write expressions for both the number of passengers and the fare per passenger in terms of \( x \), as follows:

- the fare per passenger is \( 7 + 0.10x \)
- the number of passengers is \( 2000 - 40x \).

Since the number of passengers must be at least 1600, \( 2000 - 40x \geq 1600 \), and \( x \leq 10 \), and since the number of passengers cannot exceed 2600, \( 2000 - 40x \leq 2600 \), and \( x \geq -15 \).

The domain is \(-15 \leq x \leq 10\).
The revenue function is
\[ R(x) = (7 + 0.10x)(2000 - 40x) = -4x^2 - 80x + 14000. \]
From a practical standpoint, we also require that \( x \) be an integer, in order that the fare only varies by increments of $0.10. We do not wish to consider fares that are other than multiples of 10 cents.

Therefore the problem is now to find the absolute maximum value of the revenue function.

\[ R(x) = (7 + 0.10x)(2000 - 40x) \]
\[ = -4x^2 - 80x + 14000 \]
on the interval \( -15 \leq x \leq 10 \), where \( x \) must be an integer.

\[ R'(x) = -8x - 80 \]
\( R'(x) = 0 \) when \( -8x - 80 = 0 \) and \( x = -10 \)
\( R'(x) \) is never undefined. The only extreme value for \( R \) occurs at \( x = -10 \) which is in the domain. To determine the maximum revenue, we evaluate

\[ R(-15) = -4(-15)^2 - 80(-15) + 14000 \]
\[ = 14\,300 \]
\[ R(-10) = -4(-10)^2 - 80(-10) + 14000 \]
\[ = 14\,400 \]
\[ R(10) = -4(10)^2 - 80(10) + 14000 \]
\[ = 12\,800 \]

Therefore, the maximum revenue occurs when there are \(-10\) fare increases of $0.10 each, or a fare decrease of \( 10(0.10) = $1.00 \). At a fare of $6.00, the daily revenue is $14 400, and the number of passengers is \( 2000 - 40(-10) = 2400 \).

**EXAMPLE 2**  
**Solving a problem to minimize cost**

A cylindrical chemical storage tank with a capacity of 1000 m\(^3\) is to be constructed in a warehouse that is 12 m by 15 m with a height of 11 m. The specifications call for the base to be made of sheet steel that costs $100/m\(^2\), the top to be made of sheet steel that costs $50/m\(^2\), and the wall to be made of sheet steel costing $80/m\(^2\).

a. Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it is possible, state the restrictions on the radius.
b. If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions and that minimize the cost of steel for construction.

All calculations should be accurate to two decimal places.
Solution

a. The radius of the tank cannot exceed 6 m, and the maximum height is 11 m. The volume, using \( r = 6 \) and \( h = 11 \), is \( V = \pi r^2 h = 1244 \). It is possible to build a tank with a volume of 1000 m³. There are limits on the radius and the height. Clearly, \( 0 < r \leq 6 \). Also, if \( h = 11 \), then \( \pi r^2 (11) \geq 1000 \), so \( r \geq 5.38 \). The tank can be constructed to fit in the warehouse. Its radius must be \( 5.38 \leq r \leq 6 \).

b. If the height is \( h \) metres and the radius is \( r \) metres, then the cost of the base is \( $100(\pi r^2) \) the cost of the top is \( $50(\pi r^2) \) the cost of the wall is \( $80(2\pi rh) \) The cost of the tank is \( C = 150\pi r^2 + 160\pi rh \).
Here we have two variable quantities, \( r \) and \( h \). However, since \( V = \pi r^2 h = 1000 \), \( h = \frac{1000}{\pi r^2} \).
Substituting for \( h \), we have a cost function in terms of \( r \).

\[
C(r) = 150\pi r^2 + 160\pi r \left( \frac{1000}{\pi r^2} \right)
\]
or \( C(r) = 150\pi r^2 + \frac{160000}{r} \)

From part a., we know that the domain is \( 5.38 \leq r \leq 6 \). For extreme points, set \( C'(r) = 0 \).

\[
300\pi r - \frac{160000}{r^2} = 0
\]

\[
300\pi r = \frac{160000}{r^2}
\]

\[
r^3 = \frac{1600}{3\pi}
\]

\[
r \approx 5.54
\]

This value is within the given domain, so we use the algorithm for finding maximum and minimum values.

\[
C(5.38) = 150\pi (5.38)^2 + \frac{160000}{5.38} \approx 43380
\]

\[
C(5.54) = 150\pi (5.54)^2 + \frac{160000}{5.54} \approx 43344
\]

\[
C(6) = 150\pi (6)^2 + \frac{160000}{6} \approx 43631
\]
The minimal cost is $43,344 with a tank of radius 5.54 m and a height of \( \frac{1000}{\pi(5.54)^2} = 10.37 \) m.

In summary, when solving real-life optimization problems, there are often many factors that can affect the required functions and their domains. Such factors may not be obvious from the statement of the problem. We must do research and ask many questions to address all of the factors. Solving an entire problem is a series of many steps, and optimization using calculus techniques is only one step in determining a solution.

**IN SUMMARY**

**Key Ideas**
- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced or sold.
- Economic situations usually involve minimizing costs and maximizing profits.

**Need to Know**
- In order to maximize revenue, we can use the revenue function: revenue = total revenue from the sale of \( x \) units = (price per unit) \( \times x \)
- You must always consider practical constraints, as well as mathematical constraints, when constructing your model.

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**Exercise 3.4**

**PART A**

1. The cost, in dollars, to produce \( x \) litres of maple syrup for the Elmira Maple Syrup Festival is \( C(x) = 75(\sqrt{x} - 10), \ x \geq 400. \)
   
   a. What is the average cost of producing 625 L?
   
   b. The marginal cost is \( C'(x) \), and the marginal revenue is \( R'(x) \). What is the marginal cost at 1225 L?
   
   c. How much production is needed to achieve a marginal cost of $0.50/L?
2. A sociologist determines that a foreign-language student has learned 
\[ N(t) = 20t - t^2 \] vocabulary terms after \( t \) hours of uninterrupted study.

a. How many terms are learned between times \( t = 2 \) h and \( t = 3 \) h?
b. What is the rate in terms per hour at which the student is learning at time \( t = 2 \) h?
c. What is the maximum rate in terms per hour at which the student is learning?

3. A researcher found that the level of antacid in a person’s stomach \( t \) minutes after a certain brand of antacid tablet is taken is 
\[ L(t) = \frac{6t}{t^2 + 2t + 1}. \]

a. Determine the value of \( t \) for which \( L'(t) = 0 \).
b. Determine \( L(t) \) for the value found in 3. a.
c. Using your graphing calculator, graph \( L(t) \).
d. From the graph, what can you predict about the level of antacid in a person’s stomach after 1 min?
e. What is happening to the level of antacid in a person’s stomach from \( 2 \leq t \leq 8 \) min?

PART B

4. The running cost, \( C \), in dollars per hour for an airplane cruising at a height of \( h \) metres and an air speed of 200 km/h is given by
\[ C = 4000 + \frac{h}{15} + \frac{150000000}{h} \] for the domain \( 1000 \leq h \leq 20 \ 000 \). Determine the height at which the operating cost is at a minimum, and find the operating cost per hour.

5. A rectangular piece of land is to be fenced using two kinds of fencing. Two opposite sides will be fenced using standard fencing that costs $6/m, while the other two sides will require heavy-duty fencing that costs $9/m. What are the dimensions of the rectangular lot of greatest area that can be fenced in for a cost of $9000?

6. A real estate office manages 50 apartments in a downtown building. When the rent is $900 per month, all the units are occupied. For every $25 increase in rent, one unit becomes vacant. On average, each unit requires $75 in maintenance and repairs each month. How much rent should the real estate office charge to maximize profits?

7. A bus service carries 10 000 people daily between Ajax and Union Station, and the company has space to serve up to 15 000 people per day. The cost to ride the bus is $20. Market research shows that, if the fare increases by $0.50, 200 fewer people will ride the bus. What fare should be charged to get the maximum revenue, given that the bus company must have at least $130 000 in fares a day to cover operating costs.
8. The fuel cost per hour for running a ship is approximately one half the cube of the speed plus additional fixed costs of $216 per hour. Find the most economical speed to run the ship for a 500 nautical mile trip. Note: Assume there are no major disturbances, such as heavy tides or stormy seas.

9. A 20000 m$^3$ rectangular cistern is to be made from reinforced concrete such that the interior length will be twice the height. If the cost is $40/m^2$ for the base, $100/m^2$ for the side walls, and $200/m^2$ for the roof, find the interior dimensions (correct to one decimal place) that will keep the cost to a minimum. To protect the water table, the building code specifies that no excavation can be more than 22 m deep. It also specifies that all cisterns must be at least 1 m in depth.

10. The cost of producing an ordinary cylindrical tin can is determined by the materials used for the wall and the end pieces. If the end pieces are twice as expensive per square centimetre as the wall, find the dimensions (to the nearest millimetre) to make a 1000 cm$^3$ can at minimal cost.

11. Your neighbours operate a successful bake shop. One of their specialties is a very rich whipped-cream-covered cake. They buy the cakes from a supplier who charges $6.00 per cake, and they sell 200 cakes weekly at $10.00 each. Research shows that profit from the cake sales can be increased by increasing the price. Unfortunately, for every increase of $0.50 cents, cake sales will drop by 7.
   a. What is the optimal retail price for a cake in order to obtain a maximum weekly profit?
   b. The supplier, unhappy with reduced sales, informs the owners that, if they purchase fewer than 165 cakes weekly, the cost per cake will increase to $7.50. Now what is the optimal retail price per cake, and what is the bake shop’s total weekly profit?
   c. Situations like this occur regularly in retail trade. Discuss the implications of reduced sales with increased total profit versus greater sales with smaller profits. For example, a drop in the number of customers could mean fewer sales of associated products.

12. Sandy will make a closed rectangular jewellery box with a square base from two different woods. The wood for the top and bottom costs $20/m^2$. The wood for the sides costs $30/m^2$. Find the dimensions that minimize the wood costs for a volume of 4000 cm$^3$.

13. An electronics store is selling personal CD players. The regular price for each CD player is $90. During a typical two weeks, the store sells 50 units. Past sales indicate that, for every $1 decrease in price, the store sells five more units during two weeks. Calculate the price that will maximize revenue.
14. A professional basketball team plays in an arena that holds 20,000 spectators. Average attendance at each game has been 14,000. The average ticket price is $75. Market research shows that, for each $5 reduction in the ticket price, attendance increases by 800. Find the price that will maximize revenue.

15. Through market research, a computer manufacturer found that $x$ thousand units of its new laptop will sell at a price of $2000 - 5x$ dollars per unit. The cost, $C$, in dollars of producing this many units is $C(x) = 15,000,000 + 1,800,000x + 75x^2$. Determine the level of sales that will maximize profit.

**PART C**

16. If the cost of producing $x$ items is given by the function $C(x)$, and the total revenue when $x$ items are sold is $R(x)$, then the profit function is $P(x) = R(x) - C(x)$. Show that the profit function has a critical point when the marginal revenue equals the marginal cost.

17. A fuel tank is being designed to contain 200 m$^3$ of gasoline, but the maximum length of a tank that can be safely transported to clients is 16 m long. The design of the tank calls for a cylindrical part in the middle, with hemispheres at each end. If the hemispheres are twice as expensive per unit area as the cylindrical wall, find the radius and height of the cylindrical part so the cost of manufacturing the tank will be minimal. Give the answer correct to the nearest centimetre.

18. A truck crossing the prairies at a constant speed of 110 km/h gets gas mileage of 8 km/L. Gas costs $1.15/L. The truck loses 0.10 km/L in fuel efficiency for each km/h increase in speed. Drivers are paid $35/h in wages and benefits. Fixed costs for running the truck are $15.50/h. If a trip of 450 km is planned, what speed will minimize operating expenses?

19. During a cough, the diameter of the trachea decreases. The velocity, $v$, of air in the trachea during a cough may be modelled by the formula $v(r) = Ar^2(r_0 - r)$, where $A$ is a constant, $r$ is the radius of the trachea during the cough, and $r_0$ is the radius of the trachea in a relaxed state. Find the radius of the trachea when the velocity is the greatest, and find the associated maximum velocity of air. Note that the domain for the problem is $0 \leq r \leq r_0$. 
CHAPTER 3: MAXIMIZING PROFITS

A construction company has been offered a build–operate contract for $7.8 million to construct and operate a trucking route for five years to transport ore from a mine site to a smelter. The smelter is located on a major highway, and the mine is 3 km into the bush off the road.

Construction (capital) costs are estimated as follows:
- Upgrading the highway (i.e., repaving) will be $200 000/km.
- A new gravel road from the mine to the highway will cost $500 000/km.

Operating conditions are as follows:
- There will be 100 return trips each day for 300 days a year for each of the five years the mine will be open.
- Operating costs on the gravel road will be $65/h, and average speed will be 40 km/h.
- Operating costs on the highway will be $50/h, and average speed will be 70 km/h.

Use calculus to determine if the company should accept the contract and determine the distances along the paved and gravel roads that produce optimum conditions (maximum profit). What is the maximum profit? Do not consider the time value of money in your calculations.
Review Exercise

1. Determine $f'$ and $f''$, if $f(x) = x^4 - \frac{1}{x^4}$.

2. For $y = x^9 - 7x^3 + 2$, find $\frac{d^2y}{dx^2}$.

3. Determine the velocity and acceleration of an object that moves along a straight line in such a way that its position is $s(t) = t^2 + (2t - 3)^{\frac{1}{2}}$.

4. Determine the velocity and acceleration as functions of time, $t$, for $s(t) = t - 7 + \frac{5}{t}, t \neq 0$.

5. A pellet is shot into the air. Its position above the ground at any time, $t$, is defined by $s(t) = 45t - 5t^2$ m. For what values of $t, t \geq 0$, is the upward velocity of the pellet positive? zero? negative? Draw a graph to represent the velocity of the pellet.

6. Determine the maximum and minimum of each function on the given interval.
   a. $f(x) = 2x^3 - 9x^2, -2 \leq x \leq 4$
   b. $f(x) = 12x - x^3, x \in [-3, 5]$
   c. $f(x) = 2x + \frac{18}{x}, 1 \leq x \leq 5$

7. A motorist starts braking for a stop sign. After $t$ seconds, the distance, in metres, from the front of the car to the sign is $s(t) = 62 - 16t + t^2$.
   a. How far was the front of the car from the sign when the driver started braking?
   b. Does the car go beyond the stop sign before stopping?
   c. Explain why it is unlikely that the car would hit another vehicle that is travelling perpendicular to the motorist’s road when the car first comes to a stop at the intersection.

8. The position function of an object that moves in a straight line is $s(t) = 1 + 2t - \frac{8}{t^2 + 1}, 0 \leq t \leq 2$. Calculate the maximum and minimum velocities of the object over the given time interval.

9. Suppose the cost, in dollars, of manufacturing $x$ items is approximated by $C(x) = 625 + 15x + 0.01x^2$, for $1 \leq x \leq 500$. The unit cost (the cost of manufacturing one item) would then be $U(x) = \frac{C(x)}{x}$. How many items should be manufactured in order to ensure the unit cost is minimized?
10. For each of the following cost functions, find, in dollars
   i. the cost of producing 400 items
   ii. the average cost of each of the first 400 items produced
   iii. the marginal cost when \( x = 400 \), as well as the cost of producing the
       401st item
   a. \( C(x) = 3x + 1000 \)
   b. \( C(x) = 0.004x^2 + 40x + 8000 \)
   c. \( C(x) = \sqrt{x} + 5000 \)
   d. \( C(x) = 100x^{-\frac{1}{2}} + 5x + 700 \)

11. Find the production level that minimizes the average cost per unit for the cost
    function \( C(x) = 0.004x^2 + 40x + 16000 \). Show that it is a minimum by
    using a graphing calculator to sketch the graph of the average cost function.

12. a. The position of an object moving along a straight line is described by the
    function \( s(t) = 3t^2 - 10 \) for \( t \geq 0 \). Is the object moving away from, or
    toward, its starting position when \( t = 3 \)?
    b. Repeat the problem using \( s(t) = -t^3 + 4t^2 - 10 \) for \( t \geq 0 \).

13. A particle moving along a straight line will be \( s \) centimetres from a fixed
    point at time \( t \) seconds, where \( t > 0 \) and \( s = 27t^3 + \frac{16}{t} + 10 \).
    a. Determine when the velocity will be zero.
    b. Is the particle accelerating? Explain.

14. A box with a square base and no top must have a volume of 10 000 cm³. If
    the smallest dimension is 5 cm, determine the dimensions of the box that
    minimize the amount of material used.

15. An animal breeder wishes to create five adjacent rectangular pens, each with
    an area of 2400 m². To ensure that the pens are large enough for grazing, the
    minimum for either dimension must be 10 m. Find the dimensions for the
    pens in order to keep the amount of fencing used to a minimum.

16. You are given a piece of sheet metal that is twice as long as it is wide, and the
    area of the sheet is 800 square metres. Find the dimensions of the rectangular
    box that would contain a maximum volume if it were constructed from this
    piece of metal. The box will not have a lid. Give your answer correct to one
    decimal place.

17. A cylindrical can is to hold 500 cm³ of apple juice. The design must take into
    account that the height must be between 6 cm and 15 cm, inclusive. How
    should the can be constructed so a minimum amount of material will be used
    in the construction? (Assume there will be no waste.)
18. In oil pipeline construction, the cost of pipe to go under water is 60% more than the cost of pipe used in dry-land situations. A pipeline comes to a 1 km wide river at point A and must be extended to a refinery, R, on the other side, 8 km down a straight river. Find the best way to cross the river so the total cost of the pipe is kept to a minimum. (Answer to one decimal place.)

19. A train leaves the station at 10:00 p.m. and travels due north at a speed of 100 km/h. Another train has been heading due west at 120 km/h and reaches the same station at 11:00 p.m. At what time were the two trains closest together?

20. A store sells portable MP3 players for $100 each and, at this price, sells 120 MP3 players every month. The owner of the store wishes to increase his profit, and he estimates that, for every $2 increase in the price of MP3 players, one less MP3 player will be sold each month. If each MP3 player costs the store $70, at what price should the store sell the MP3 players to maximize profit?

21. An offshore oil well, P, is located in the ocean 5 km from the nearest point on the shore, A. A pipeline is to be built to take oil from P to a refinery that is 20 km along the straight shoreline from A. If it costs $100 000 per kilometre to lay pipe underwater and only $75 000 per kilometre to lay pipe on land, what route from the well to the refinery will be the cheapest? (Give your answer correct to one decimal place.)

22. The printed area of a page in a book will be 81 cm². The margins at the top and bottom of the page will each be 3 cm deep. The margins at the sides of the page will each be 2 cm wide. What page dimensions will minimize the amount of paper?

23. A rectangular rose garden will be surrounded on three sides by a brick wall and by a fence on the fourth side. The area of the garden will be 1000 m². The cost of the brick wall is $192/m. The cost of the fencing is $48/m. Find the dimensions of the garden so that the cost of the materials is as small as possible.

24. A boat leaves a dock at 2:00 P.M., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 P.M. When were the boats closest to each other?

25. Two towns, Ancaster and Dundas, are 4 km and 6 km, respectively, from an old railroad line that has been made into a bike trail. Points C and D on the trail are the closest points to the two towns, respectively. These points are 8 km apart. Where should a rest stop be built to minimize the length of new trail that must be built from both towns to the rest stop?
26. Find the absolute maximum and minimum values.
   a. \( f(x) = x^2 - 2x + 6, \ -1 \leq x \leq 7 \)
   b. \( f(x) = x^3 + x^2, \ -3 \leq x \leq 3 \)
   c. \( f(x) = x^3 - 12x + 2, \ -5 \leq x \leq 5 \)
   d. \( f(x) = 3x^5 - 5x^3, \ -2 \leq x \leq 4 \)

27. Sam applies the brakes steadily to stop his car, which is travelling at 20 m/s. The position of the car, \( s \), in metres at \( t \) seconds is given by \( s(t) = 20t - 0.3t^3 \).
   Determine
   a. the stopping distance   b. the stopping time   c. the deceleration at 2 s

28. Calculate each of the following.
   a. \( f''(2) \) where \( f(x) = 5x^3 - x \)
   b. \( f''(-1) \) where \( f(x) = -2x^3 + x^2 \)
   c. \( f''(0) \) where \( f(x) = (4x - 1)^4 \)
   d. \( f''(1) \) where \( f(x) = \frac{2x}{x - 5} \)
   e. \( f''(4) \) where \( f(x) = \sqrt{x + 5} \)
   f. \( f''(8) \) where \( f(x) = \sqrt[3]{x} \)

29. An object moves along a straight line. The object’s position at time \( t \) is given by \( s(t) \). Find the position, velocity, acceleration, and speed at the specified time.
   a. \( s(t) = \frac{2t}{t + 3}; \ t = 3 \)
   b. \( s(t) = t + \frac{5}{t + 2}; \ t = 1 \)

30. The function \( s(t) = (t^2 + t)^\frac{3}{2}, \ t \geq 0 \), represents the displacement, \( s \), in metres of a particle moving along a straight line after \( t \) seconds.
   a. Determine \( v(t) \) and \( a(t) \).
   b. Find the average velocity during the first five seconds.
   c. Determine the velocity at exactly 5 s.
   d. Find the average acceleration during the first five seconds.
   e. Determine the acceleration at exactly 5 s.
1. Determine the second derivative of each of the following.
   a. \( y = 7x^2 - 9x + 22 \)  
   b. \( f(x) = -9x^5 - 4x^3 + 6x - 12 \)  
   c. \( y = 5x^{-3} + 10x^3 \)  
   d. \( f(x) = (4x - 8)^3 \)

2. For each of the following displacement functions, calculate the velocity and acceleration at the indicated time.
   a. \( s(t) = -3t^3 + 5t^2 - 6t \) when \( t = 3 \)  
   b. \( s(t) = (2t - 5)^3 \) when \( t = 2 \)

3. The position function of an object moving horizontally along a straight line as a function of time is \( s(t) = t^2 - 3t + 2 \), \( t \geq 0 \), in metres, at time, \( t \), in seconds.
   a. Determine the velocity and acceleration of the object.
   b. Determine the position of the object when the velocity is 0.
   c. Determine the speed of the object when the position is 0.
   d. When does it move to the left?
   e. Determine the average velocity from \( t = 2 \) s to \( t = 5 \) s.

4. Determine the maximum and minimum of each function on the given interval.
   a. \( f(x) = x^3 - 12x + 2 \), \( -5 \leq x \leq 5 \)  
   b. \( f(x) = x + \frac{9}{x} \), \( x \in [1, 6] \)

5. After a football is punted, its height, \( h \), in metres above the ground at \( t \) seconds can be modelled by \( h(t) = -4.9t^2 + 21t + 0.45 \), \( t \geq 0 \).
   a. When does the ball reach its maximum height?
   b. What is the maximum height?

6. A man purchased 2000 m of used wire fencing at an auction. He and his wife want to use the fencing to create three adjacent rectangular paddocks. Find the dimensions of the paddocks so the fence encloses the largest possible area.

7. An engineer working on a new generation of computer called The Beaver is using compact VLSI circuits. The container design for the CPU is to be determined by marketing considerations and must be rectangular in shape. It must contain exactly 10 000 cm\(^3\) of interior space, and the length must be twice the height. If the cost of the base is $0.02/cm\(^2\), the cost of the side walls is $0.05/cm\(^2\), and the cost of the upper face is $0.10/cm\(^2\), find the dimensions to the nearest millimetre that will keep the cost of the container to a minimum.

8. The landlord of a 50-unit apartment building is planning to increase the rent. Currently, residents pay $850 per month, and all the units are occupied. A real estate agency advises that every $100 increase in rent will result in 10 vacant units. What rent should the landlord charge to maximize revenue?