QUESTION 14 Page 24

Have students use sequences of differences and graphing to distinguish between the arithmetic and quadratic sequences.

Answers

14. (a) *example*: arithmetic since the common difference is 2
    (b) *example*: quadratic because all terms are square numbers
    (c) *example*: quadratic because the graph appears to be part of a parabola
    (d) *example*: arithmetic because each term is found by adding $\frac{3}{4}$ to the previous term
    (e) *example*: neither quadratic nor arithmetic; it is a cubic sequence
    (f) *example*: quadratic sequence because $D_2 = -6$

QUESTIONS 15 AND 16 Page 25

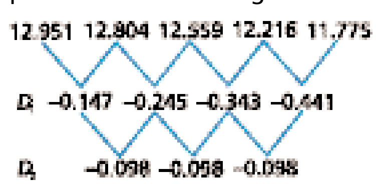
For Question 15, students will probably draw pictures of where they are and where they are going. For example, after 1 s, the truck is 80 m from the bridge. After 2 s, the truck is 60 m from the bridge. Less formal answers should be accepted from students. Make sure Question 15 is assigned before Question 16.

For Question 16, encourage students to find differences to three decimal places. You may need to discuss this process with students, and how this is like Question 15 and how this is different from Question 15.

Answers

15. (a) The truck has travelled 20 m after 1 s, so it is $100 - 20 = 80$ m away from the bridge after 1 s. Similarly, the truck has travelled 40 m after 2 s, and 60 m after 3 s. So, the truck is $100 - 40 = 60$ m away after 2 s, and is $100 - 60 = 40$ m away after 3 s.
    (b) The sequence {80, 60, 40, 20, 0} is arithmetic, because each term in D_1 is -20.
    (c) The relation between the distance, *d,* in metres from the truck to the bridge and the time, *t,* in seconds is $d = 100 - 20t$. When $d = 0$, $t = 5$. The truck passes under the bridge after 5 s.

16.



   (a) $D_1 = \{-0.147, -0.245, -0.343, -0.441, ...\}$ and $D_2 = \{-0.098, -0.098, -0.098, ...\}$; quadratic
   (b) The next term in D_1 is $-0.441 + (-0.098) = -0.539$, so the next term in the original sequence is $11.775 + (-0.539) = 11.236$. The original sequence is quadratic, since all the terms in D_2 are the same.

Investigation 6

Cubic Number Patterns

[Suggested time: 60 min] ***[Text page 25]***

Purpose

Students will explore the properties of power sequences up to and including cubic sequences. They will see how the degree or greatest exponent of a relation (polynomial) is related to the sequences of differences.

Management Suggestions and Materials

Give groups of students cube-a-links so that they can make the first few cubes.

Procedure

Steps A and B

Once students identify the first four terms in the sequence as cubic numbers, they can extend the pattern to 10 terms: {1, 8, 27, 64, 125, 216, 343, 512, 729, 1000}. Have them compare the sequence to a sequence of square numbers to see that cubic numbers grow much more rapidly. A graph of $y = x^3$ and $y = x^2$ shows the difference in growth rates.

Step C

A spreadsheet can be used to find the sequences of differences D_1, D_2, and D_3 quickly.

	A	B	C	D	E	F	G	H	I	J	K
1	Term number	1	2	3	4	5	6	7	8	9	10
2	Cubic sequence	1	8	27	64	125	216	343	512	729	1000
3	D_1		7	19	37	61	91	127	169	217	271
4	D_2			12	18	24	30	36	42	48	54
5	D_3				6	6	6	6	6	6	6

Students can see that the third-level difference, D_3, is a constant. They should know that the sequence is neither arithmetic nor quadratic because neither D_1 nor D_2 is constant.

Investigation Questions

QUESTION 17 Page 26

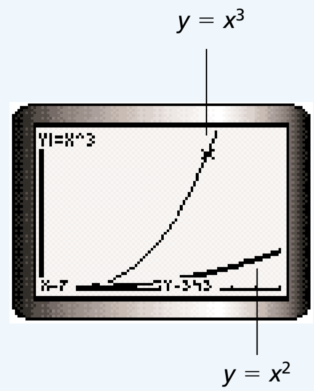
Show students a square pyramid by making one of table-tennis balls or marbles glued together. Students can stack circular objects such as coins to represent cannon balls.

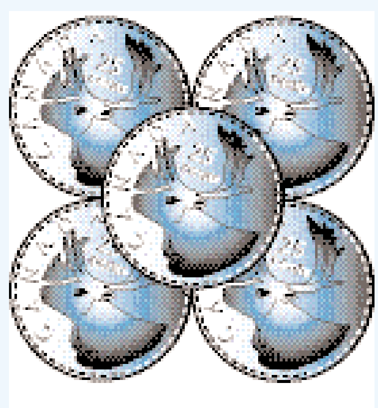
Make sure students realize that the next three pyramids will have a square bottom layer with dimensions 5 × 5, 6 × 6, and 7 × 7, respectively. Therefore, the fifth pyramid will have 30 + 25 = 55 cannon balls, the sixth pyramid will have 55 + 36 = 91 cannon balls, and the seventh pyramid will have 91 + 49 = 140

Think about ....

The Diagrams

Have students note that the numbers of smaller cubes in the first four larger cubes are 1, 8, 27, and 64, respectively. Each of these numbers can be written as a cube: 1^3, 2^3, 3^3, and 4^3.





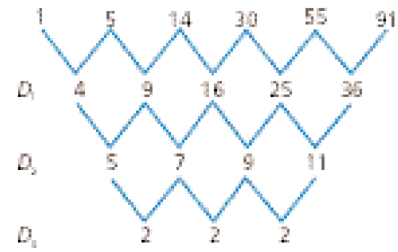
A pyramid of 4 + 1 = 5 coins

Think about …

Question 17

Make sure students realize that each layer of cannon balls is the shape of a square, although some students might object that the top cannon ball is not in the form of a square. If so, mention that they should consider it a 1×1 square. This means that the number of cannon balls can be shown as $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$, where n is the number of layers in the pyramid. The fourth pyramid has a floor of 4×4 or 4^2 or 16 cannon balls. Adding another layer of 5×5 or 5^2 will increase the number by 25 cannon balls and, therefore, the fifth pyramid will have $30 + 25 = 55$ cannon balls. Students can use first- and second-level differences from Question 17(b) to show that the sequence is neither arithmetic nor quadratic.

cannon balls. This forms a sequence of seven terms: {1, 5, 14, 30, 55, 91, 140}.



Students should note that D_3 is constant. Draw their attention to the cubic sequence in the Investigation and have them predict what type of relation will have a constant D_3.

Answers

17. (a) {1, 5, 14, 30, 55, 91, 140}
    (b) D_3
    (c) a cubic relation or rule

QUESTION 18 Page 26

Give students examples of arithmetic and quadratic sequences to help them generalize.

- An arithmetic sequence can be expressed as $t_n = 3n + 2$, where the greatest exponent is 1.
- A quadratic sequence can be expressed as $t_n = 2n^2 + n$, where the greatest exponent is 2.

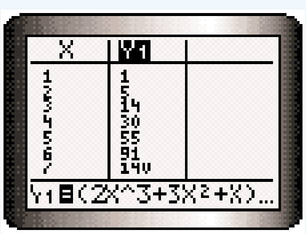
An arithmetic sequence has D_1 as a constant, and a quadratic sequence has D_2 as a constant. Ask the students to compare the exponents (degree) of the relations and the subscripts in the first- and second-level differences, and ask them to think about the form of a relation whose sequence has a constant D_3.

Answer

18. a cubic relation because the greatest exponent is 3

QUESTION 19 Page 26

Have students use a graphing calculator to build the sequence. They can enter the relation into the Y= editor and see the results in TABLE, as shown in the margin.

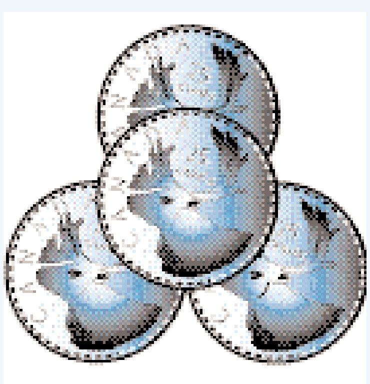


Answers

19. (a) The relation is called cubic because the greatest exponent or degree is 3, which is often referred to as a "cube."
    (b) See the table in the margin.
    (c) $t_{50} = \frac{2(50)^3 + 3(50)^2 + 50}{6} = 42\,925$

QUESTION 20 Page 26

Students could again use circular objects such as coins to model triangular pyramids.



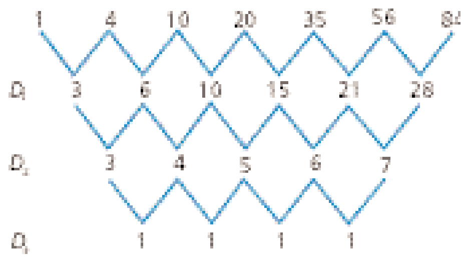
A triangular pyramid of 3 + 1 = 4 coins

Have them refer to the triangular numbers in Question 9 after Investigation 5. They can see that triangles are the layers of a triangular pyramid. For example, the triangular number sequence is {1, 3, 6, 10, 15, 21, 28, …}. Therefore, the number of apples in the first seven pyramids must be as follows.

Pyramid 1 1
Pyramid 2 1 + 3 = 4
Pyramid 3 1 + 3 + 6 = 10
Pyramid 4 1 + 3 + 6 + 10 = 20
Pyramid 5 1 + 3 + 6 + 10 + 15 = 35
Pyramid 6 1 + 3 + 6 + 10 + 15 + 21 = 56
Pyramid 7 1 + 3 + 6 + 10 + 15 + 21 + 28 = 84

The sequence is {1, 4, 10, 20, 35, 56, 84}.

The constant term of 1 in the third-level sequence of differences shows that the sequence is a cubic sequence.



Answers

20. (a) to (d) See the preceding discussion.

QUESTION 21 Page 27

Answers

21. (a) {16.89, 135.09, 455.95, 1080.76, 2110.86, 3647.56, 5792.19}
    (b) D_1 {118.20, 320.86, 624.81, 1030.10, 1536.70, 2144.63}
    D_2 {202.66, 303.95, 405.29, 506.60, 607.93}
    D_3 {101.29, 101.34, 101.31, 101.33}; D_3
    (c) cubic because D_3 is approximately the constant 101.32
    (d) A cubic relationship exists between the circumference and the amount of helium needed for a spherical balloon. Similar cubic relationships exist between the radius or the diameter and the volume or capacity of the spherical balloon.

Check Your Understanding

[Completion and discussion: 45 min]

QUESTION 22 Page 27

Mention that each sequence generated by a quadratic or cubic relation has one term that is of the form $y = ax^n$, where n is the greatest power or exponent. Thus, sequences generated by such relations are called *power sequences*.

Students can check the answer to part (b) by building sequences from simple relations such as $t_n = n^4$. They will see that eventually a sequence of differences becomes a constant number. A spreadsheet will speed up and simplify the calculations.

Answers

22. (a) See the previous discussion.
    (b) If the greatest exponent or degree of a power relation (polynomial) is n, then D_n is a constant number.

Challenge Yourself

Tell students that the 10 people were trying to set a record for the *Guinness Book of World Records*.

They can use TABLE for the function presented in Question 19 to find the value of n that gives a value of 4900 cans.

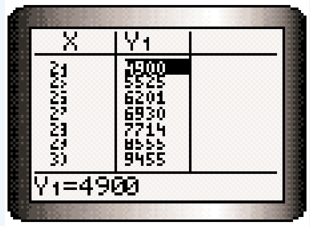


TABLE shows that a pyramid with 24 layers would use 4900 cans.

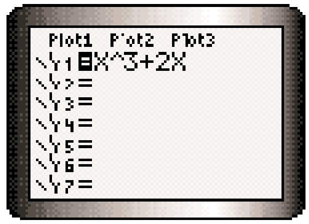
Think about …

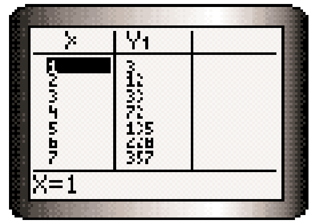
Sequences Generated by Cubic Relations

After these experiences, students should realize that D_3 is constant for a cubic relation whose degree or greatest exponent is also 3.

QUESTION 23 Page 27

Students can use the Y= editor of a graphing calculator to build a power sequence. The table can be shown to another student, who will try to name the type of relation behind the sequence.





X is the term number, and Y_1 is the sequence generated by the cubic relation $Y_1 = X^3 + 2X$.

Answer

23. Answers will vary.

Challenge Yourself

It will be helpful if students use a spreadsheet or the LIST editor on a graphing calculator for the calculations.

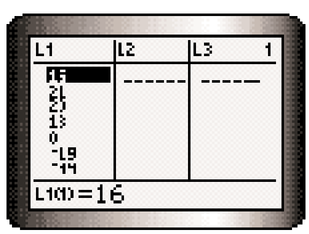
The following spreadsheets show a sample of the results that might be expected.

(a) In this table, two arithmetic sequences were generated by the linear relations $t_n = 3n + 1$ and $t_n = -n + 5$. The fourth row shows the product of the two arithmetic sequences multiplied together.

	A	B	C	D	E	F	G	H	I	J
1	Term number	1	2	3	4	5	6	7	8	9
2	$t_n = 3n + 1$	4	7	10	13	16	19	22	25	28
3	$t_n = -n + 5$	4	3	2	1	0	−1	−2	−3	−4
4	Product	16	21	20	13	0	−19	−44	−75	−112
5	D_1		5	−1	−7	−13	−19	−25	−31	−37
6	D_2			−6	−6	−6	−6	−6	−6	−6

Teach students how to use the ΔList(feature on a graphing calculator, a process that subtracts the first element in a list from the second element, subtracts the second element from the third element, and so forth).

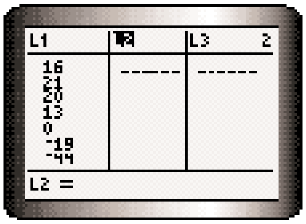
Students can enter the product of the sequences under a LIST such as L_1:



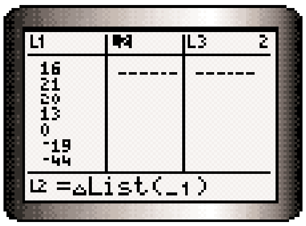
Management Tip

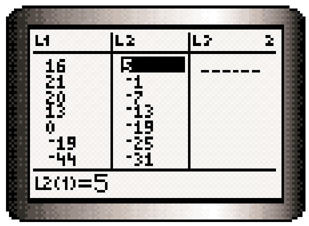
Students do not need to use a spreadsheet to complete this Challenge Yourself.

Next, the cursor can be moved to the top of L_2:

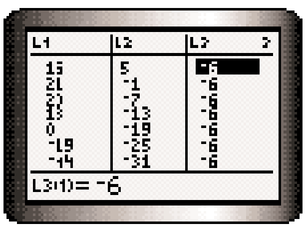


After entering 2nd LIST and selecting OPS and ΔList(, and entering 2nd L_1, the sequence of first-level differences will be entered under L_2:





A similar procedure can be used to create the sequence of second-level differences under LIST L_3:



Students can see that the product of two arithmetic sequences is a quadratic sequence because D_2 is a constant.

(b) It will greatly simplify the activity to use the same spreadsheet but change one linear relation to a quadratic (leaving the other linear relation).

	A	B	C	D	E	F	G	H	I	J
1	Term number	1	2	3	4	5	6	7	8	9
2	$t_n = 3n + 1$	4	7	10	13	16	19	22	25	28
3	$t_n = n^2 + 5$	6	9	14	21	30	41	54	69	86
4	Product	24	63	140	273	480	779	1188	1725	2408
5	D_1		39	77	133	207	299	409	537	683
6	D_2			38	56	74	92	110	128	146
7	D_3				18	18	18	18	18	18

As might be expected, the product of a linear relation and a quadratic relation is a cubic sequence because D_3 is a constant 18.

QUESTION 24 Page 28

Give students grid paper and tracing paper (hamburger-patty paper works well). They can draw each grid and then trace onto the tracing paper squares that are 2-by-2, 3-by-3, and so on. The tracing paper superimposed on the grid will help students to count the number of squares on each grid.

Have students note that the sequence is identical to the sequence from the situation in which cannon balls were arranged in a square pyramid. Thus, the same sequence appears in a different context. Students can extend the pattern of first-level differences to find the number of squares in an 8-by-8 chessboard.

Answers

24. (a) and (b) {1, 5, 14, 30, 55, 91}
    (c) It is a cubic relationship because D_3 is constant.
    (d) 204

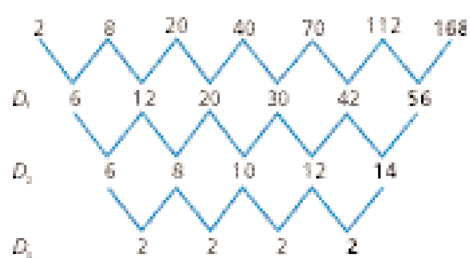
QUESTIONS 25 AND 26 Page 28

Have students refer to and use the sequences from Question 13 and Investigation 6 to develop each of the two sequences.

Sequence for oblong figures: {2, 6, 12, 20, 30, 42, …}
Sequence for cubic numbers: {1, 8, 27, 64, 125, 216, …}

Total of dots by adding dots in oblong figures: {2, 8, 20, 40, 70, 112}
Total of small cubes by adding larger cubes: {1, 9, 36, 100, 225, 441}

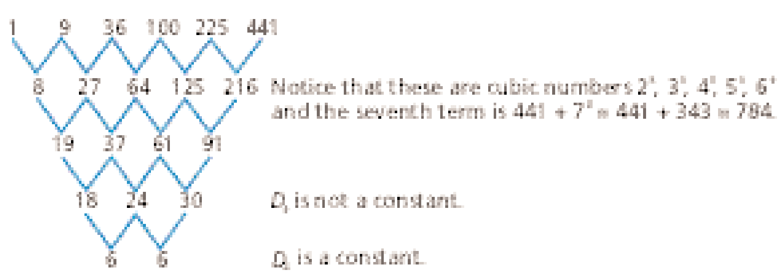
Students find the seventh term of the oblong sequence by working backward from D_3.



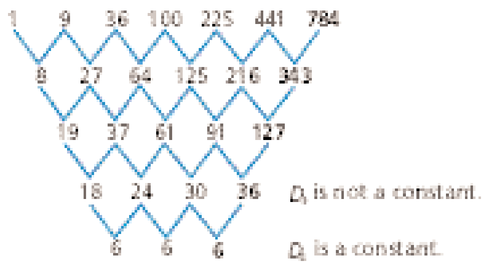
Some students might note that the sequence of cubes contains square numbers: $\{1^2, 3^2, 6^2, 10^2, 15^2, 21^2\}$. They can use this square pattern to reason that the seventh term must be

$(21 + 7)^2 = 28^2$ or 784.

They can also extend various patterns in the sequences of differences to find the seventh term.



Assist students in working backward from D_4 to find the next term in the sequence.



Answers

25. (a) {2, 8, 20, 40, 70, 112, …}
    (b) cubic because D_3 is constant
    (c) 168
26. (a) {1, 9, 36, 100, 225, 441, …}
    (b) another type of sequence; likely a power sequence whose greatest exponent is 4
    (c) 784

Investigation 7

Patterns of Rapid Growth

[Suggested time: 60 min] ***[Text page 29]***

Purpose

Students will explore geometric sequences and compare them to other sequences they have looked at. Although students should be expected to learn the term *geometric sequence* and list some properties, the term *common ratio* is not expected.

Management Tip

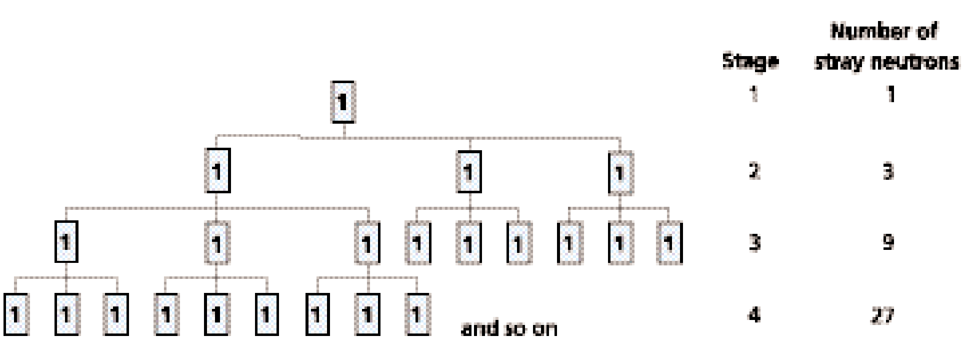
Have students look again at the geometric sequences in this Investigation after they begin the study of exponential relations presented in Chapter 3, Exponential Growth.

Procedure

Begin the Investigation with a class discussion on nuclear fission and the rapid growth of particles.

Step A

Suggest that students make a "family tree" similar to their solutions of the Fibonacci rabbit problem.



Step B

Have students look at the diagram to see that each neutron releases three other neutrons at each stage of a chain reaction. So, the number of stray neutrons at each of the 10 stages is shown by the sequence {1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19 683, …}. Students should note that there have been 9 splits during the 10 stages.

Students should realize that, during an uncontrolled nuclear chain reaction, an astronomical number of particles are released within a fraction of a second. Mention that if one kilogram of uranium underwent fission, it would release the same amount of energy as burning 3000 tonnes of coal or exploding 9000 tonnes of TNT.

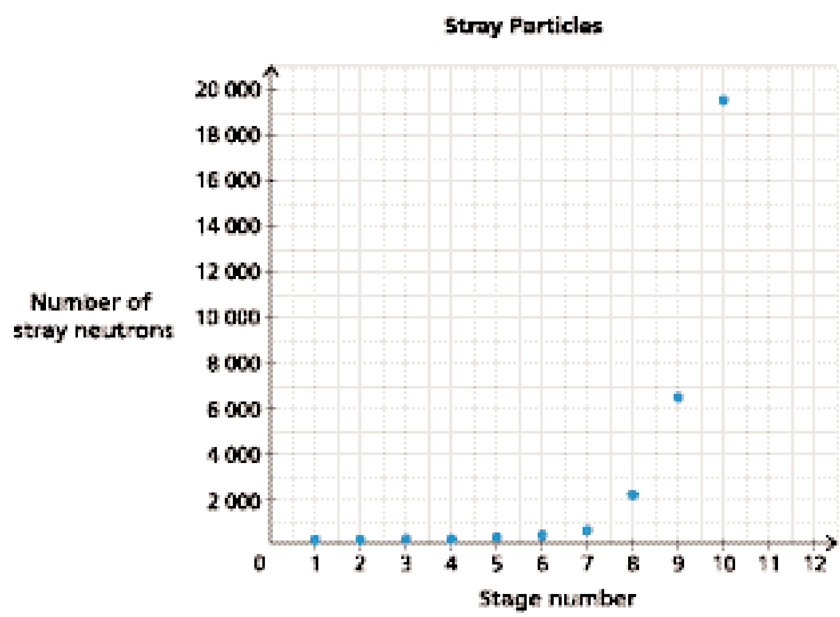
Students might see the following number patterns:

- Each number is odd.
- The numbers of stray neutrons increase rapidly.
- Each number is a power of 3.
- Each successive number is 3 times as great as the previous number.
- Each previous number is $\frac{1}{3}$ of the next number.

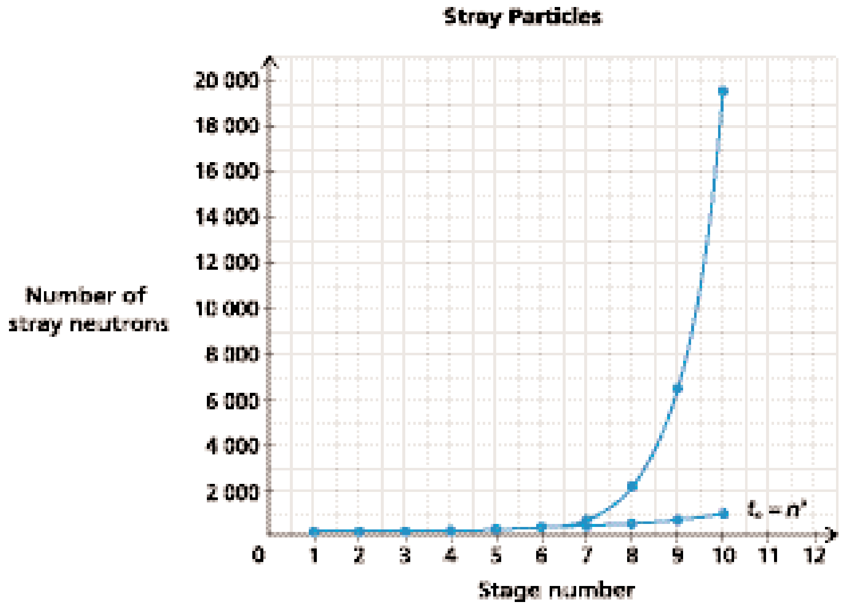
Step C

The graph shows that the number of stray particles increases rapidly in very few stages. The curve initially appears relatively flat and then increases dramatically in height from the seventh stage onward. Students can see from the shape of the graph that there is a non-linear relationship between the number of stages and the number of stray particles. Therefore, the sequence in Step A is not an arithmetic sequence.

Stage number	Number of stray neutrons
1	1
2	3
3	9
4	27
5	81
6	243
7	729
8	2 187
9	6 561
10	19 683



For comparison, have students graph a quadratic sequence (such as $t_n = n^2$) or a cubic sequence (such as $t_n = n^3$).



The graph shows that the neutron sequence grows much more quickly than a cubic sequence, and is likely some other type of sequence.

Steps D and E

The following table lists the first-, second-, and third-level differences of the sequence in Step B.

Stage number	1	2	3	4	5	6	7	8	9	10
Number of Particles	**1**	3	9	27	81	243	729	2187	6561	19 683
D_1		**2**	6	18	54	162	486	1458	4374	13 122
D_2			**4**	12	36	108	324	972	2916	8748
D_3				**8**	24	72	216	648	1944	5832

Some students might see that each sequence in the table could be extended by repeatedly multiplying the previous term by a constant: in this case, 3.

Students should also note that, because no sequence of differences has a constant term or common difference, the nuclear reaction sequence is not an example of an arithmetic, quadratic, or cubic sequence.

Some students might want to build the fourth- and fifth-level sequences of differences to see if the sequence is a form of power sequence. However, explain that the sequence is a special sequence whose sequences of differences will *never* have a common term.

This sequence is known as a geometric sequence, and is built by multiplying each previous term by the same number. For example, the sequence {2, 10, 50, 250, ...} is built by multiplying the previous term by 5. The formal definition of a geometric sequence is presented in Question 28.

Step F

Make sure students see that, in each sequence in Steps B and D, every term comes from multiplying the previous term by 3. Point out that the original sequence can be rewritten as $\{3^0, 3^1, 3^2, ...\}$, for example. Show students that none of these have

Note

Students will likely note that in this sequence, *each term can be multiplied by the same number to get the next term in the sequence*. The formal definition of a geometric sequence is presented in Question 28.

common differences, but it is still a sequence. Students can generate the sequence simply by multiplying each subsequent term by three.

Step G

Dividing each term in the original sequence {1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19 683} by the previous term gives the sequence {3, 3, 3, 3, 3, 3, 3, 3, 3, 3}. Students should note that the new sequence has a constant term of 3 because the number of neutrons at one stage is three times the number at a previous stage. This relationship exists because each neutron releases three other neutrons at the next stage of the reaction.

Investigation Questions

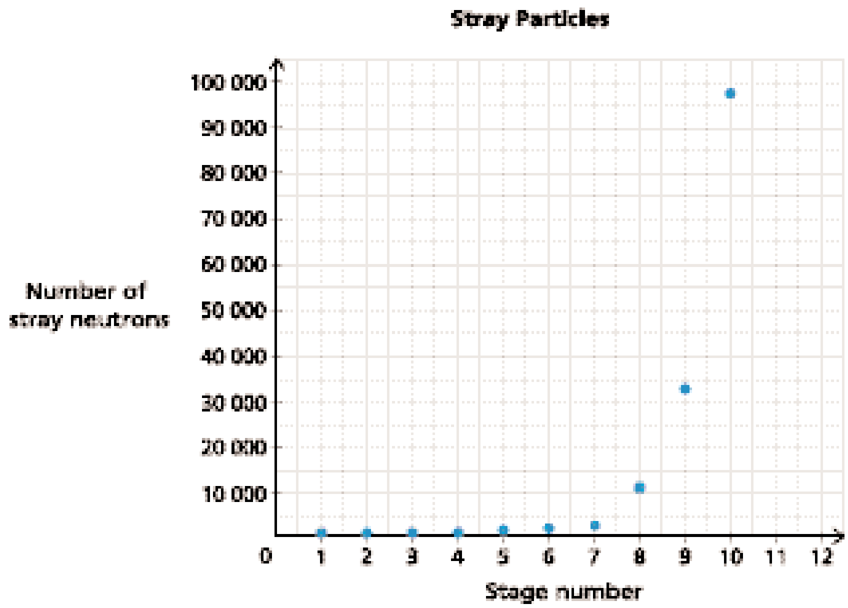
QUESTION 27 Page 30

Students should see that the number patterns in this sequence are like those found in the Investigation. The initial five neutrons each release three neutrons, all of which release a further three neutrons at each successive stage. So, the sequence is multiples of powers of three and, therefore, the sequence in part (b) will have a constant or common ratio 3.

The resulting sequence and sequences of differences are shown in the following table. Students can see several number patterns in the table; for example, each sequence is created by repeatedly multiplying the previous term by 3. Once again, this sequence (like the one in Step B) is neither an arithmetic, quadratic, nor cubic sequence, but another type of sequence: a geometric sequence.

Stage	1	2	3	4	5	6	7	8	9	10
Stray particles	5	15	45	135	405	1215	3645	10 935	32 805	98 415
D_1		10	30	90	270	810	2430	7290	21 870	65 610
D_2			20	60	180	540	1620	4860	14 580	43 740
D_3				40	120	360	1080	3240	9720	29 160

Note that each y-value on the graph is five times higher than the y-values of the graph drawn in Step C because each stage contains five times as many stray particles as in the original reaction.



Stage number	Number of stray neutrons
1	5
2	15
3	45
4	135
5	405
6	1 215
7	3 645
8	10 935
9	32 805
10	98 415

Dividing each term in the original sequence {5, 15, 45, 135, 405, 1215, 3645, 10 935, 32 805, 98 415, ...} by the previous term gives the sequence {3, 3, 3, 3, 3, 3, 3, 3, 3, 3, ...}. Students should note that the new sequence has a constant term of 3 because the number of stray neutrons at one stage is three times the number at a previous stage. This relationship exists because each neutron releases three other neutrons at the next stage of the reaction.

Answers

27. See the previous discussion.

QUESTIONS 28 AND 29 Page 30

Answers

28. (a) {4, 20, 100, 500, 2500, 12 500, 62 500, 312 500, 1 562 500, 7 812 500}
    (b) The sequence is geometric since each term is five times the previous term.
29. Sequences will vary, but each will be a geometric sequence.

Check Your Understanding

[Completion and discussion: 30 min]

QUESTION 30 Page 30

Help students find the number of *E. coli* bacteria created in one day by using a spreadsheet to complete a table such as the one on the following page.

Notebook Entry

Have students write down the definition of a geometric sequence with the example and at least one more example of their own.

geometric sequence—a number sequence that is built by always multiplying by the same number; for example, the sequence {2, 6, 18, 54, ...} is built by multiplying the previous term by 3

Management Tip

Have students research and report on the sources of *E. coli* bacteria and how human beings can protect themselves from this dangerous "bug." Students might check the following Web site for more information about the geometric or exponential growth of bacteria: **www.ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/andromed.html** Students should discuss what assumptions are being made about the growth of the bacteria: in this case, no bacteria die.

Divisions	1	2	3	4	5	6	7	8	9	10	11	12
Time (h)			1			2			3			4
Bacteria	2	4	8	16	32	64	128	256	512	1024	2048	4096
Divisions	13	14	15	16	17	18	19	20	21	22	23	24
Time (h)			5			6			7			8
Bacteria	8192	16 384	32 768	65 536	131 072	262 144	524 288	1 048 576	2 097 152	4 194 304	8 388 608	16 777 216
Divisions	25	26	27	28	29	30	31	32	33	34	35	36
Time (h)			9			10			11			12
Bacteria	33 554 432	67 108 864	1.34E+08	2.68E+08	5.37E+08	1.07E+09	2.15E+09	4.29E+09	8.59E+09	1.72E+10	3.44E+10	6.87E+10
Divisions	37	38	39	40	41	42	43	44	45	46	47	48
Time (h)			13			14			15			16
Bacteria	1.37E+11	2.75E+11	5.5E+11	1.1E+12	2.2E+12	4.4E+12	8.8E+12	1.76E+13	3.52E+13	7.04E+13	1.41E+14	2.81E+14
Divisions	49	50	51	52	53	54	55	56	57	58	59	60
Time (h)			17			18			19			20
Bacteria	5.63E+14	1.13E+15	2.25E+15	4.5E+15	9.01E+15	1.8E+16	3.6E+16	7.21E+16	1.44E+17	2.88E+17	5.76E+17	1.15E+18
Divisions	61	62	63	64	65	66	67	68	69	70	71	72
Time (h)			21			22			23			24
Bacteria	2.31E+18	4.61E+18	9.22E+18	1.84E+19	3.69E+19	7.38E+19	1.48E+20	2.95E+20	5.9E+20	1.18E+21	2.36E+21	4.72E+21

Management Tip

In the table above, a number like 1.34E+08 means 1.34×10^8, or 134 000 000. You may need to discuss this notation with students.

Encourage students to work together to complete this question.

Answers

30. (a) {1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096}
    (b) geometric sequence, since each term is two times the previous term
    (c) approximately 4.72×10^{21}

QUESTION 31 Page 31

Repeatedly reduce a 10-by-10 grid to help the students see that the area of each copy is 70% of the area of the previous copy. Students can answer part (d) by extending the sequence until the value of one term is less than 1.

Answers

31. (a) 70 cm^2 and 49 cm^2
    (b) {70, 49, 34.30, 24.01, 16.81, 11.76, 8.24, 5.76, 4.04, 2.82}
    (c) geometric sequence, since each term is 0.7 times the previous term
    (d) The 13th copy has an area of 0.97 cm^2.

QUESTION 32 Page 31

Have students research and report on both simple interest and compound interest investments and how they differ.

Make sure students understand that adding 5% interest to the previous value of an investment is the same as multiplying the previous value by 105%, or 1.05.

Value in Year 2 is 5% of \$1050 plus 100% of \$1050
Value in Year 2 is $0.05 \times \$1050$ plus $1 \times \$1050$
Value in Year 2 is $(0.05 \times \$1050) + (1 \times \$1050) = \$1050 \times (1 + 0.05)$
$= 1.05\ (\$1050)$

The constant term is 1.05 so the sequence is geometric.

Answers

32. (a) geometric because each term is multiplied by 1.05

    (b) \$1628.89

QUESTION 33 Page 31

Students can repeatedly multiply the initial population of 356 000 by 1.01 until a term exceeds 400 000.

Answer

33. in about 11 or 12 years

Chapter Project

Plant and Tree Growth

This is the conclusion of the Chapter Project. Tell students that the amount of woody growth in a tree each year depends on how well its leaves function and how much nourishment they manufacture. Forestry experts measure tree diameter at DBH (diameter breast height, about 4.5 feet or about 1.37 m). The thickness of the growth ring depends on the tree and local resources and conditions.

Students can complete this part of the Chapter Project as they are doing the Chapter Review and Case Studies.

(a) Students should see that the diameter increases by 2 cm and not 1 cm because the annual growth ring encloses the tree on all sides. The sequence is {17, 19, 21, 23, 25, 27, 29, 31, 33, 35}.

(b) and (c) Refer to the circumference and area formulas in the note in the Student Book.

The sequence for circumferences rounded to two decimal places is {53.41, 59.69, 65.97, 72.26, 78.54, 84.82, 91.11, 97.39, 103.67, 109.96}.

The sequence for cross-sectional areas rounded to two decimal places is {226.98, 283.53, 346.36, 415.48, 490.87, 572.56, 660.52, 754.77, 855.30, 962.11}.

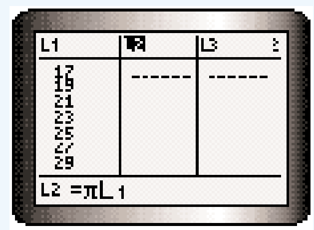
(d) Students can use reasoning, graphing, or sequences of differences to name each sequence.

Because 2 cm is added to the previous year's diameter of the tree, the sequence in part (a) must be an arithmetic sequence.

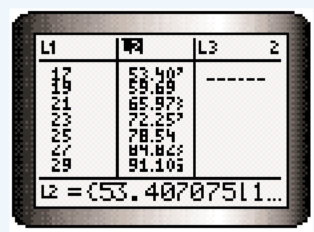
The graph of the circumference versus the term number shows a linear relationship, and the sequence in part (b) must be an arithmetic sequence.

Technology

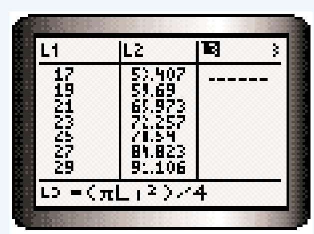
If the sequence of diameters is entered into list L_1 in the List editor of a graphing calculator, the formulas for finding the circumference and area can be entered into L_2 and L_3 as shown:

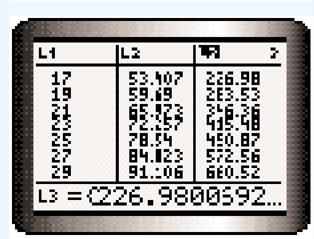


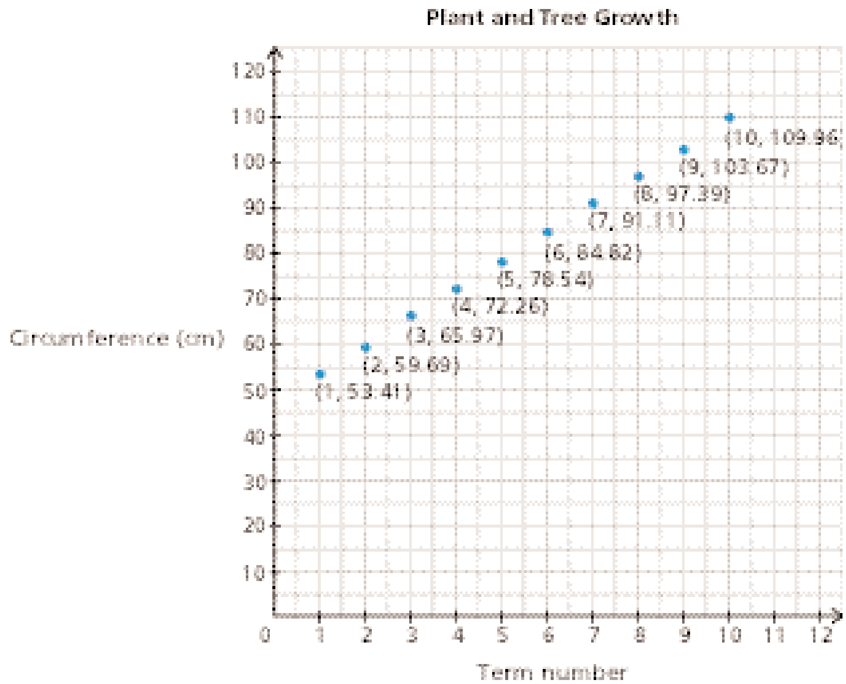
When ENTER is pressed, the calculations for the circumference for each diameter in L_1 appear in L_2:



In the same way, the area for each diameter can be calculated and placed in list L_3:







The sequence of second-level differences shows that the sequence for part (c) involving cross-sectional area is a quadratic sequence:

D_1 {56.55, 62.83, 69.12, 75.40, 81.68, 87.96, 94.25, 100.53, 106.81}

D_2 {6.28, 6.28, 6.28, 6.28, 6.28, 6.28, 6.28, 6.28}

Students should also see that each formula for circumference and area could be used to name each type of sequence.

- For a circle with diameter d, the formula $C = \pi d$ can be used to find the circumference. Because π is a constant number, there must be a linear relationship between d and C and, therefore, the sequence of circumferences must be an arithmetic sequence.
- For a circle with diameter d, the formula $A = \frac{1}{4}\pi d^2$ can be used to find the area. Because π and the fraction $\frac{1}{4}$ are constant numbers and the diameter is squared, there must be a quadratic relationship between d and A. Therefore, the sequence of areas must be a quadratic sequence.

PUTTING IT TOGETHER

Case Study 1: Jet Fuel Consumption

Performance Expectations

Students who successfully complete this Case Study will demonstrate the ability to:

- understand growth patterns that are linear and quadratic
- use sequences of differences and graphing to distinguish between arithmetic and quadratic sequences
- extend patterns to find rules or relations that build a sequence
- use the slope of a graph and D_1 to find the rule for building an arithmetic sequence
- use patterns to solve real-world problems involving linear and quadratic relationships

Use

This Case Study can be used in a variety of ways:

- as a potential portfolio item assigned at the end of this chapter
- as a homework project assigned as part of the end-of-chapter review
- as the basis for a student presentation or display project
- as a quiz or test item at the end of the chapter

Management

Students can work **individually** or in **pairs**. Students can use graphing calculators.

Answers

(a) {215 733, 215 721, 215 709, 215 697, 215 685, 215 673, 215 661, 215 649, 215 637, 215 625}

(b) Students can use graphing and first-level common differences to determine that the sequence in part (a) is an arithmetic sequence. However, because each term for the remaining fuel is 12 L more than the next term, most students should realize that it is an arithmetic sequence with a constant sequence of first-level differences with a term of −12.

(c) *example*: Repeatedly subtract 12 from 215 745 or, more formally, $t_n = 215\,733 - 12(n - 1)$
$= 215\,745 - 12n$

(d) Students can substitute 50 000 for t_n in the rule or equation in part (c) to solve the problem:

$$\begin{aligned} t_n &= 215\,745 - 12n \\ 50\,000 &= 215\,745 - 12n \\ n &= \frac{165\,745}{12} \\ &\doteq 13\,812.083\,33 \end{aligned}$$

The plane will travel about 13 812 km before it has 50 000 L of fuel remaining.

(e) The sequence is a quadratic sequence because $D_2 = -2000$.

(f) In calculating D_1, students should note that the sequence is {−3000, −5000, …, −19 000, …}. This recursive pattern can be used to continue the sequence:
{116 733 − 21 000 = 95 733,
95 733 − 23 000 = 72 733,
72 733 − 25 000 = 47 733,
47 733 − 27 000 = 20 733,
20 733 − 29 000 = −8267}.
Therefore, the plane will run out of fuel between 14 and 15 km into the flight.

Extension

Answers

(a) The jet burns fuel at a rate of 12 L/km and the cost is $0.50/L; therefore, the sequence would be {$6, $12, $18, $24, $30, $36, $42, $48, $54, $60}.

(b) arithmetic sequence because of the common difference of $6

(c) $t_n = 6n$

(d) $6 × 12 791 = $76 746

Assessment

You can use the following scale to assign a score to each part of the student work:

0 not done
1 major errors or omissions
2 some minor errors
3 complete and correct

Case Study 2: Tiling Costs

Performance Expectations

Students who successfully complete this Case Study will demonstrate the ability to:

- understand quadratic growth patterns
- use first- and second-level differences and graphing to distinguish between arithmetic and quadratic sequences
- use first- and second-level differences to extend a sequence
- use patterns to solve problems with real-world linear and quadratic relationships
- use patterns to find a rule to build simple quadratic sequences

Use

This Case Study can be used in a variety of ways:

- as a potential portfolio item assigned at the end of this chapter
- as a homework project assigned as part of the end-of-chapter review
- as a quiz or test item at the end of the chapter

Management

Students can work **individually** or in **pairs**. They will need grid paper to draw the tiling patterns and to make graphs. A graphing calculator is also useful.

Answers

(a) Cost = \$89

\$5	\$5	\$5	\$5	\$5
\$5	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$5
\$5	\$5	\$5	\$5	\$5

Cost = \$116

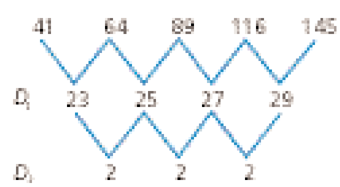
\$5	\$5	\$5	\$5	\$5	\$5
\$5	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$5
\$5	\$5	\$5	\$5	\$5	\$5

Cost = \$145

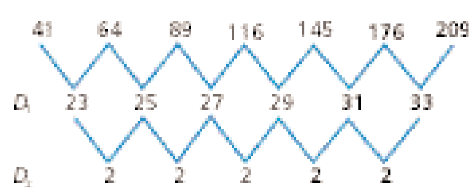
\$5	\$5	\$5	\$5	\$5	\$5	\$5
\$5	\$1	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$1	\$5
\$5	\$1	\$1	\$1	\$1	\$1	\$5
\$5	\$5	\$5	\$5	\$5	\$5	\$5

(b) Each cost has been written as a term in a sequence: {\$41, \$64, \$89, \$116, \$145}.

(c) The sequences of differences show that the sequence is a quadratic sequence.



(d) As shown below, the first-level differences can be used to extend the bottom row of the table. The sixth tiling arrangement must cost \$145 + \$31 = \$176, and the seventh tiling arrangement must cost \$176 + \$33 = \$209.



Extension

Students can replace the costs of the tiles in the original arrangements of tiling patterns to find the new costs: {\$85, \$140, \$205, \$280, \$365}.

D_1 {55, 65, 75, 85}
D_2 {10, 10, 10}

The sequence is a quadratic sequence because the second-level sequence of differences has a constant term of 10.

Assessment

You can use the following scale to assign a score to each part of the student work:

0 not done
1 major errors or omissions
2 some minor errors
3 complete and correct

PRACTICE

Answers, Text pages 42–44

Question 1

(a) {5, 3, 8, 6, 11, 9, 14, 12, 17, 15}

(b) *example*: The numbers alternate by "less 2" and "plus 5."

(c) *example*: The odd-numbered terms differ by 3: {5, 8, 11, 14, 17, 20, 23, …}.

(d) *example*: The even-numbered terms are multiples of 3: {3, 6, 9, 12, 15, 18, 21, …}.

(e) *example*: Bus Stop 16 is an even number, and even numbers have numbers of passengers that are multiples of 3. Bus Stop 16 is the eighth multiple of 3 so 24 passengers will be on the bus.

(f) If 24 passengers were at Bus Stop 16, Bus Stop 32 should have 48 passengers.

Question 2

(a) {10, 16, 22}

(b) The next matchstick figure has 28 matches and the sequence is {10, 16, 22, 28}.

(c) *example*: The terms differ by a constant 6.

(d) *example*: The first matchstick figure contains 10 matches. The 100th figure has 99 sets of 6 matches added to the figure with 10 matches. The total would be $10 + (99 \times 6) = 604$ matches.

Question 3

(a) *Example*: The number of ways is the sum of the two previous numbers of ways; it is similar to a Fibonacci sequence.

(b) 5 can be formed $5 + 3 = 8$ ways; 6 can be formed $8 + 5 = 13$ ways; and 7 can be formed $8 + 13 = 21$ ways.

(c) similar to a Fibonacci sequence, except for the initial two terms

Question 4

(a) *example*: Each sequence has a constant difference between terms.

(b) *examples*: The sequence in the column {1, 10, 19, 28, 37, 46, 55} is an arithmetic sequence whose rule is *Start with 1 and repeatedly add 9* or, more formally, $t_n = 1 + (n - 1) \times 9 = 9n - 8$.

The sequence along the diagonal, {1, 11, 21, 31, 41, 51, 61}, is an arithmetic sequence whose rule is *Start with 1 and repeatedly add 10* or, more formally, $t_n = 1 + (n - 1) \times 10 = 10n - 9$.

The sequence in the row {1, 2, 3, 4, 5, 6, 7, 8, 9} is an arithmetic sequence whose rule is *Start with 1 and repeatedly add 1* or $t_n = 1 + (n - 1) \times 1 = n$.

Question 5

(a) *example*: The sequence {4, 11, 18, ...} is arithmetic because the first-level difference, D_1, is 7.

(b) *example: Start with 4 and then add on 49 7s or $4 + 49 \text{ x } 7 = 347$*; more formally,

$$t_n = 4 + (n - 1) \times 7$$
$$= 7n - 3;$$
$$t_{50} = (7 \times 50) - 3$$
$$= 347$$

(c) $1000 = 7n - 3$; $\frac{1003}{7} = n \doteq 143.2857$. So $t_{143} = 998$ and $t_{144} = 1005$ is slightly greater than 1000.

Question 6

(a) The sequence is {5, 13, 21, 29, 37, …}, and it is made by adding 8 to each previous number. It is, therefore, an arithmetic sequence.

(b) $\$5 + (19 \times \$8) = \$157$

(c) The rule is $t_n = 5 + (n - 1) \times 8 = 8n - 3$ or $t_n = 8n - 3$. Substituting \$1000 for t_n, we can solve for n to find the tower number: $1000 = 8n - 3$; $n = \frac{1003}{8} = 125.375$. Therefore, the 125th tower will be slightly less than \$1000 and the 126th tower will be slightly greater than \$1000.

(d) Students can either draw the graph or reason that because the constant difference is 8, the graph will be a straight line with a slope of 8.

(e) The sequence would be {10, 26, 42, 58, 74, …}, which is still an arithmetic sequence with $D_1 = 16$; the 20th tower would have a value of $\$10 + (19 \times \$16) = \$314$; the 63rd tower would be the first to have a value of more than \$1000; the graph would be a straight line with a slope of 16.

Question 7

(a) *example*: {4, 6, 8, 10, 12, ...}: Each ring of the doorbell gives a constant increase of two guests; therefore, the sequence is arithmetic.

(b) *example*: On the first ring, 4 guests arrived; after 9 more rings, $9 \times 2 = 18$ more guests arrived and the total number of guests is 22; more formally, $t_n = 4 + (n - 1) \times 2$ or the total number of guests is $4 + (10 - 1) \times 2 = 22$.

Question 8

(a) *example*: He will do 20 sit-ups plus 29×3 sit-ups on the 30th day; that is, $20 + (30 - 1) \times 3 = 107$ sit-ups.

(b) *example*: Establish the rule *total sit-ups* $= 20 + (n - 1) \times 3$; therefore, $500 = 20 + 3n - 3$; $500 = 3n + 17$; $3n = 483$; $n = 161$ days. 161 days produces 500 sit-ups; 162 days produces 503 sit-ups.

Question 9

(a) {6, 10, 14}

(b) *example*: Each time a block is added, six square faces are added but two are hidden, for a total gain of four square faces. Therefore, the sequence will grow by 4 for each block that is added. {6, 10, 14, 18, 22, 26, 30, 34, 38, 42}

(c) an arithmetic sequence with a constant difference of 4

(d) $t_n = 6 + (n - 1) \times 4$
$= 4n + 2$

(e) $t_{20} = (4 \times 20) + 2$
$= 82$ squares to be painted

(f) $t_{24} = (4 \times 24) + 2$
$= 98$ squares to be painted

Question 10

(a) not arithmetic

(b) arithmetic because the constant increase is 1

(c) arithmetic because the constant increase is 2

(d) not arithmetic

(e) arithmetic because the constant increase is 2

(f) arithmetic because D_1 is -5

Question 11

The sequence in 10(b), {51, 52, 53, 54, 55, 56, …}, is built by the rule $t_n = n + 50$ and $t_{50} = 50 + 50 = 100$.

The sequence in 10(c), {−9, −7, −5, −3, −1, 1, 3, …}, is built by the rule $t_n = 2n - 11$ and $t_{50} = 100 - 11 = 89$.

The sequence in 10(e), $\left\{\frac{1}{2}, 2\frac{1}{2}, 4\frac{1}{2}, 6\frac{1}{2}, 8\frac{1}{2}, 10\frac{1}{2}, 12\frac{1}{2}, 14\frac{1}{2}, \ldots\right\}$, is built by the rule $t_n = 2n - 1\frac{1}{2}$ and $t_{50} = 2(50) - 1\frac{1}{2} = 98\frac{1}{2}$.

The sequence in 10(f), {0.3, −4.7, −9.7, −14.7, −19.7, −24.7, …}, is built by the rule $t_n = -5n + 5.3$ and $t_{50} = -5(50) + 5.3 = -244.7$.

Question 12

(a) {4, 1, −2, −5, −8, …}

(b) {6.5, 10, 13.5, 17, 20.5, …}

(c) {0, −2, −4, −6, −8, …}

(d) $\left\{\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, 3\frac{3}{4}, 4\frac{1}{2}, \ldots\right\}$

Question 13

(a) {1, 4, 9}

(b) *example*: The numbers are square numbers.

(c) *example*: The 100th larger triangle will have $100^2 = 10\,000$ smaller triangles.

Question 14

(a)

Number of teams	2	3	**4**	**5**	6	7	8	9	10
Number of games	1	3	6	**10**	15	21	28	36	45

(b) *example*: The difference between the number of games is the sequence {2, 3, 4, 5, …}, the number of games is one-half of n times $(n - 1)$, where n is the number of teams: $\frac{1}{2}(4 \times 5) = 10$. To find the number of games, you can either continue the counting sequence (adding 10 to 45 to get 55 games for 11 teams, adding 11 to 55 to get 66 games for 12 teams, and so forth), or you can use the pattern or rule:

number of games $= \frac{1}{2}(20 \times 19) = 190$ games.

(c) {1, 3, 6, 10, 15, 21, 28, 36, 45} is a quadratic sequence because $D_2 = 1$.

(d) The pattern in part (c) can be used to develop a rule: $g = \frac{n(n-1)}{2}$, where n is the number of teams and g is the number of games needed to finish a round-robin tournament.

(e) number of games for 28 teams $= 0.5(28)(27) = 378$

Question 15

(a) If a sequence has $t_1 = 3$ and the sequence of first-level differences is {2, 4, 6, 8, …}, the first six terms of the sequence are {3, 5, 9, 15, 23, 33}.

(b) The sequence of first-level differences does not have a constant term; therefore, the sequence in part (a) is not an arithmetic sequence.

(c) $D_2 = 2$ and, therefore, the sequence in part (a) is a quadratic sequence.

Question 16

Remind students that the sequence includes those people who receive the e-mail and, therefore, the person who first writes and sends the e-mail is not considered part of the sequence.

(a) The sequence is {4, 16, 64, 256, 1024, 4096}.

(b) geometric sequence, since each term is four times the previous term

Question 17

Assist with Question 17 (c) by having students write each term in the sequence as a power of 2: {1, 2, 4, 8, 16, 32, 64, …} = $\{2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \ldots\}$. The exponent in the power of 2 that is the number of grains on the 64^{th} square will be $64 - 1 = 63$.

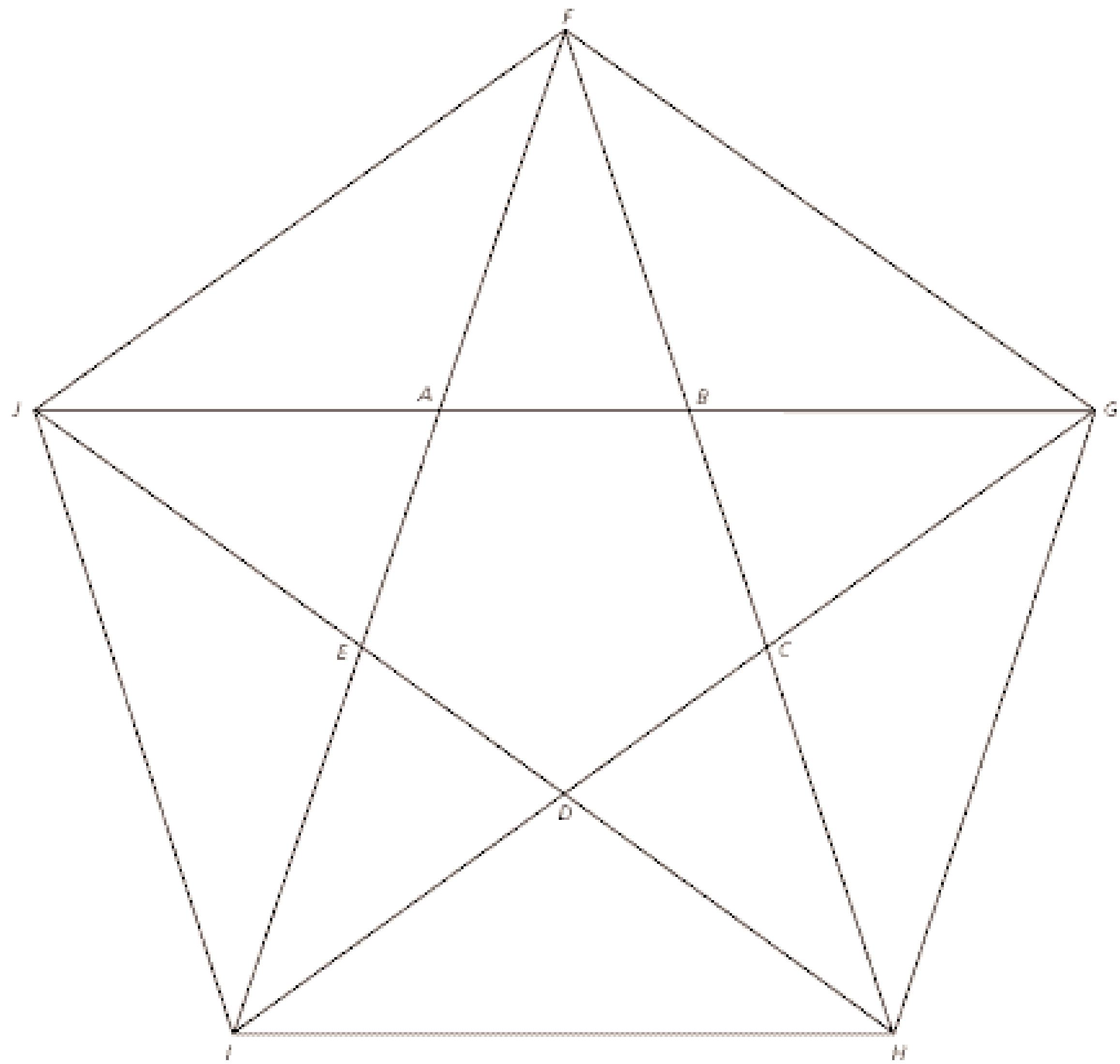
(a) {1, 2, 4, 8, 16, 32, 64, 128, 256, 512}

(b) geometric sequence; each term is twice the previous term

(c) 2^{63}, or about 9.22×10^{18}

(d) {1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19 683}; geometric sequence; 3^{63}, or about 1.14×10^{30}

PENTAGRAM



FIBONACCI NUMBERS

1	1	2	3	5	8	13	21	34	55
89	144	233	377	610	987	1597	2584	4181	6765
10 946	17 711	28 657	46 368	75 025	121 393	196 418	317 811	514 229	832 040
1 346 269	2 178 309	3 524 578	5 702 887	9 227 465	14 9303 52	24 157 817	39 088 169	63 245 986	102 334 155

1	1	2	3	5	8	13	21	34	55
89	144	233	377	610	987	1597	2584	4181	6765
10 946	17 711	28 657	46 368	75 025	121 393	196 418	317 811	514 229	832 040
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1 346 269	2 178 309	3 524 578	5 702 887	9 227 465	14 930 352	24 157 817	390 88 169	63 245 986	102 334 155

1	1	2	3	5	8	13	21	34	55
89	144	233	377	610	987	1597	2584	4181	6765
10 946	17 711	28 657	46 368	75 025	121 393	196 418	317 811	514 229	832 040
1 346 269	2 178 309	3 524 578	5 702 887	9 227 465	14 930 352	24 157 817	39 088 169	63 245 986	102 334 155

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