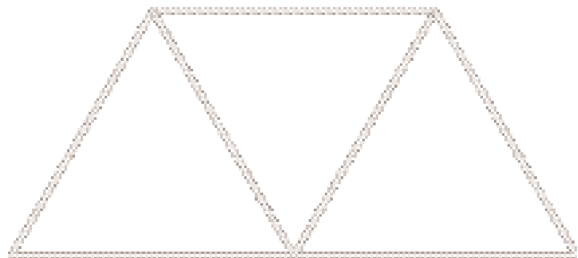
Check Your Understanding

– Note –

The diagram in Question 7 shows three equilateral triangles forming three sections of the railing.

7. A bridge railing is made of connecting sections that are equilateral triangles. The sequence {3, 5, 7, …} stands for the total number of rods in the railing as each section is added.

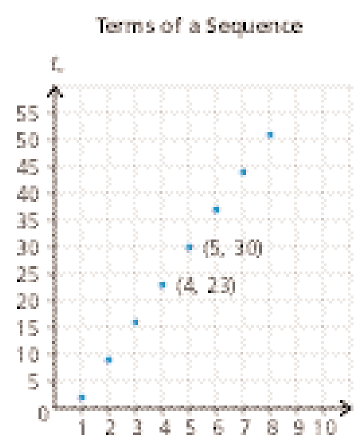


(a) Make a sequence where t_1 to t_{10} are the number of metal rods needed to build from 1 to 10 sections.

(b) For the sequence in part (a), graph the value of the term versus the term number. Use the graph or a pattern to develop a rule or relation used to create the sequence.

(c) Use the relation or rule to find the number of metal rods for a bridge railing made from 200 sections of equilateral triangles.

8. The graph represents the term numbers and the values of the terms of the same sequence. For example, the fourth term, t_4, is 23 and the fifth term, t_5, is 30.



(a) How can you tell by looking at the graph that the sequence is an arithmetic sequence?

(b) What is the slope of the graph?

(c) How does the slope of the graph relate to the common difference in the sequence?

(d) Use the common difference to find the first three terms of the sequence.

(e) Use the graph or a pattern to describe a relation that relates the value of each term to its term number.

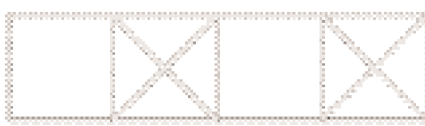
(f) Use the relation to find t_{100}, the 100^{th} term in the sequence.

9. Find out whether each sequence is arithmetic. If the sequence is arithmetic, find the common difference.
   (a) {–4, –8, –12, –16, …}
   (b) {3, 6, 9, 12, …}
   (c) {2, 7, 12, 17, …}
   (d) {1, 3, 9, 27, 81, 343, …}
   (e) {6.5, 8.5, 10.5, 12.5, 14.5, …}
   (f) {1, 4, 9, 16, 25, 36, …}
   (g) $\left\{\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, \ldots\right\}$
   (h) {24, 19, 14, 9, …}

10. For each arithmetic sequence in Question 9, describe the relation used to create the sequence. Then find t_{200}, the 200th term.

11. Explain why the Fibonacci sequence is not an arithmetic sequence.

12. A contractor alternates two shapes to form a fence.



   (a) Make a sequence that shows the number of rods needed to build a fence with one to six sections of these alternating shapes.
   (b) Is the sequence in part (a) an arithmetic sequence? How do you know?
   (c) Use a pattern or rule to find the number of rods in a fence built with 41 sections of alternating shapes.

Investigation 4

Generalizing Patterns

Jason borrowed $100 from his older sister. She wanted interest of 5% of the $100 only to be paid each month until the loan of $100 could be repaid.

Purpose

Use various methods to make rules for building arithmetic sequences.

CHALLENGE yourself

Create an arithmetic sequence in which each term of the sequence of differences is

(a) –5

(b) –3.5

(c) $-\frac{3}{4}$

(d) 150

– Note –

The diagram in Question 12 shows four sections of a fence with alternating shapes.

Think about...

The Interest and Amount Owed

Write the interest and amount owed as a sequence. Explain how you know that the sequence is an arithmetic sequence.

Procedure

A. Use blocks, cube-a-links, or grid paper to show the amount of interest in each of the first six months if no payments are made on the loan.

B. Use the results of Step A to complete the table.

Month	0	1	2	3	4	5	6
Interest ($)	0	5					
Total amount owed ($)	100	105					

C. Using the data in the table, graph the months from one to six and the total amount owed to Jason's sister in each month.

D. Find the slope of the line for the graph that you made in Step C.

Investigation Questions

13. (a) By how much does the amount owed increase each month?

(b) How does the increase in the amount owed each month relate to the slope of the graph in Step D?

(c) How does the slope of the graph relate to the common difference if the amount owed is written as a sequence?

14. (a) If you knew the number of months that Jason paid interest, how could you find the amount that he owed?

(b) Use your answer to part (a) to describe a relationship where x is the number of months that Jason paid interest and y is the total amount owed.

15. Enter the data from the table in the Investigation into your graphing calculator. One list should contain the number of months and the other list should contain the amount owed.

(a) Graph the data. Explain how the shape of the graph can help you decide which procedure to use to find the equation of the curve of best fit.

(b) Use your calculator to find the equation of the curve of best fit.

(c) Compare the equation of the curve of best fit to the rule that you created in Question 14(b). What do you notice?

16. Jason used the monthly interest or the common difference, D_1 (each constant term in the sequence of differences), to make a rule for finding the amount that he owed his sister.

$t_1 = 100 + (1 \times 5)$ The amount owed after one month

$t_2 = 100 + (2 \times 5)$ The amount owed after two months

$t_3 = 100 + (3 \times 5)$ The amount owed after three months

$t_4 = 100 + (4 \times 5)$ The amount owed after four months

$t_5 = 100 + (5 \times 5)$ The amount owed after five months

Use his method to find the amount owed after n months.

17. Use any method or rule to find the amount owed by Jason after each time period.

(a) 10 months (b) 20 months

(c) 24 months (d) 40 months

18. (a) Make a sequence that shows the amount owed at the end of each month if Jason borrowed $500 from his sister and for six months she charged him 8% monthly interest on the $500 owed.

(b) Make a rule to find the n^{th} term, t_n, where n is the number of months that interest was charged.

(c) Use your rule to find the amount Jason owes after 25 months.

Check Your Understanding

19. Use a rule or relation to generate at least six terms of an arithmetic sequence. Trade sequences with another student and make a rule to generate the arithmetic sequence that you were given.

20. Make an arithmetic sequence with at least six terms, and then make new sequences by doing the following. For each operation, start again with your original sequence.

(a) Pick any number. Add it to each term of your original sequence to get another new sequence.

(b) Pick another number. Subtract it from each term of your original sequence to get another new sequence.

(c) Pick another number. Multiply it by each term of your original sequence to get another new sequence.

(d) Pick another number. Use it to divide each term of your original sequence to get another new sequence.

(e) Square each term of your original sequence to get another new sequence.

Which of the new sequences are arithmetic sequences? How do you know? For each new sequence that is an arithmetic sequence, create a relation for finding t_n, the n^{th} term of the sequence.

Think about...

Question 18

What *annual* interest rate is Jason's sister charging?

Do you think she is being fair in charging this rate? Explain.

CHALLENGE yourself

Make an arithmetic sequence for each given fact. Create a rule that can be used to build each term in the sequence.

(a) The fifth term is 12.

(b) The first term is 1 and the sixth term is 31.

(c) The first term is a negative integer and the fifth term is a positive integer.

(d) The common difference is a negative fraction.

(e) The first term is your age and the eleventh term is 100.

Did You Know?

Bamboo grows quite differently from trees. The bamboo stem emerges from the ground as buds with the same diameter as the final stem. The bamboo grows longer in much the same way as a telescope, extending at a very rapid rate.

Dendrocalamus giganteus is one of the world's largest bamboo species. Because it can reach heights of 30 m or more and attain a thickness of 25 to 30 cm, it is often used to make water pipes, buckets, and even boat masts.

21. (a) The cost of printing is \$0.02 per page. What is the cost of printing a book that is 400 pages long? Write a relation or rule to calculate the cost of printing a book with n pages.

(b) What is the cost of making a 400-page book if the cost of binding the book is \$4 and is added to the cost of printing? Write a relation to calculate the cost of making a book with n pages?

(c) Write a relation to calculate the cost of making an n-page book if the cost of printing is \$0.03 per page and the cost of binding is \$5.

22. (a) Sonia is paid a royalty of \$2 for each CD sold by her recording company. Write a relation or rule to calculate her royalties on a sale of n CDs.

(b) Write a rule or relation to calculate her earnings if she also receives \$10 000 cash in addition to her royalties.

(c) Create a rule to calculate her earnings on the sale of n CDs with a royalty of \$2.50 per CD and cash of \$15 000.

(d) Use your rules in parts (a), (b), and (c) to calculate her earnings on a sale of 50 000 CDs.

Chapter Project

Plant and Tree Growth

Bamboo is a grass that can grow very rapidly. Suppose a 5-m-high bamboo stem has a vertical growth rate of 60 cm per day.

(a) Create a sequence to show the height of the bamboo after one to seven days of growth.

(b) Graph the height of the bamboo versus the number of days of growth.

(c) Use the graph to name the type of sequence in part (a). Find the rule used to build the sequence.

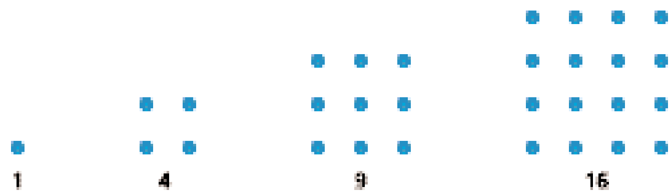
(d) Use the rule to find the number of days needed for the bamboo to reach a height of over 29 m.

1.3 Number Patterns: Part 2

Investigation 5

Solving Problems Using Sequences of Differences

Numbers that can be arranged in the shape of a square are called *square* numbers. The first four square numbers and the number of dots used to form each square are shown below.



Purpose

Explore various ways that number patterns can grow, other than at a uniform or constant rate. Use sequences of differences to solve problems.

Procedure

A. Draw the next two square numbers. Record the numbers of dots needed to draw each square.

B. Write the number of dots in each square as terms of a sequence.

C. Make a second sequence, D_1, by finding the sequence of differences between successive terms in the sequence in Step B.

D. Make a third sequence, D_2, by finding the sequence of differences between successive terms in your sequence D_1 from Step C. Describe what you notice about each term in this third sequence.

Investigation Questions

1. How do you know that the sequence in Step B is not an arithmetic sequence?

2. (a) Graph the number of dots in each square number versus the term number. Describe the shape of the graph.

   (b) How is the graph in part (a) different from the graph of an arithmetic sequence?

Did You Know?

Mathematicians in ancient Greece studied sequences made of *figurate numbers*, which are numbers that can be arranged in geometric figures or shapes.

Think about...

Step B

Look at the numbers in your sequence. What number patterns can you find?

– Note –

The results of subtracting adjacent terms in a sequence are referred to as the sequence of **first-level differences**. These differences form a new sequence, D_1. If you subtract adjacent terms in D_1, you get the sequence of **second-level differences**, D_2. Another subtraction of adjacent terms gives you the sequence of **third-level differences**, D_3.

(c) Find a rule or relation that will generate the sequence in Step B.

(d) Use the relation or rule to find the number of dots in the 50^{th} square number.

3. How do you know that the relation or rule used in Question 2(c) is a **quadratic relation or rule**?

quadratic relation or rule—a relation or rule in which the greatest value of the exponent is 2. For example, the quadratic rule $y = 2x^2 + 1$ generates the sequence {3, 9, 19, 33} for x-values of 1 to 4.

Check Your Understanding

4. (a) Use each relation below to build a sequence with 10 terms.

$t_n = 2n^2$ $\quad t_n = 2n^2 - 3$ $\quad t_n = 2n - 3$ $\quad t_n = n^3$

(b) Find the sequence of first-level differences, D_1, for each sequence in part (a). If D_1 is not a constant number, find the sequence of second-level differences, D_2.

(c) Which relations are quadratic? Why?

(d) Explain how second-level differences can be used to identify quadratic sequences.

quadratic sequence—a sequence whose terms are numbers that can be plotted to show a quadratic relation

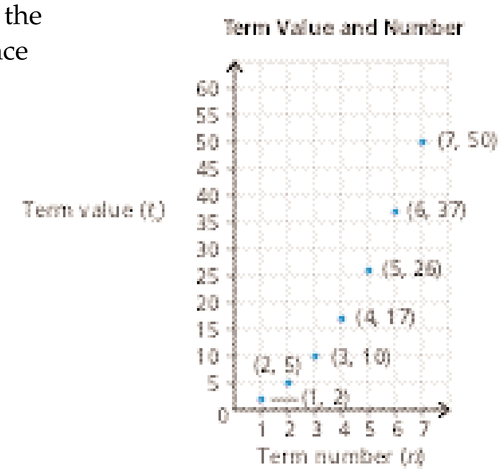
5. Use a rule to create either an arithmetic sequence or a **quadratic sequence** with at least six terms. Exchange sequences with another student. Name the type of sequence created by that student and explain how you know.

6. How are arithmetic sequences different from quadratic sequences?

7. The graph is a scatter plot of the value of the term of a sequence versus the term number.

(a) How do you know that the sequence is not arithmetic?

(b) What kind of sequence is represented by the graph? Why?



8. Which rules create an arithmetic sequence, and which create a quadratic sequence?

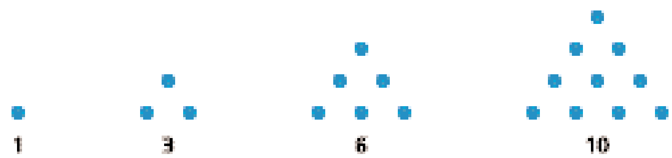
(a) Write 5, then repeatedly add 2 to it.

(b) Write 8, then repeatedly subtract 3 from it.

(c) Square each number from 1 to 10. Then add –3 to each new number.

(d) Multiply 1 by 2, 2 by 3, 3 by 4, 4 by 5, 5 by 6, and so on.

9. Numbers that can be shown in the shape of a triangle are called *triangular* numbers. The first four triangular numbers and the number of dots used to form each triangle are shown.



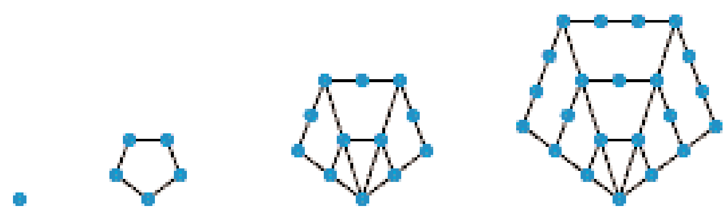
(a) Make a sequence of 10 terms, where the terms show the number of dots needed to draw the first 10 triangular numbers.

(b) Name the type of sequence created in part (a).

(c) Use a pattern to find the relationship used to build the sequence in part (a).

(d) Use the results of part (c) to find the number of dots in the 50^{th} triangular number.

10. The figures for the first four *pentagonal* numbers are shown.



(a) Count the number of dots in each pentagonal figure. Use these pentagonal numbers to make a sequence of four terms. Use a pattern to find the next term in the sequence.

(b) Use sequences of differences to name the type of sequence made in part (a).

(c) How many dots are in the fifth and sixth pentagonal figures?

11. Compare the sequences of second-level differences, D_2, for the sequences of triangular numbers, square numbers, and pentagonal numbers. Describe what you notice.

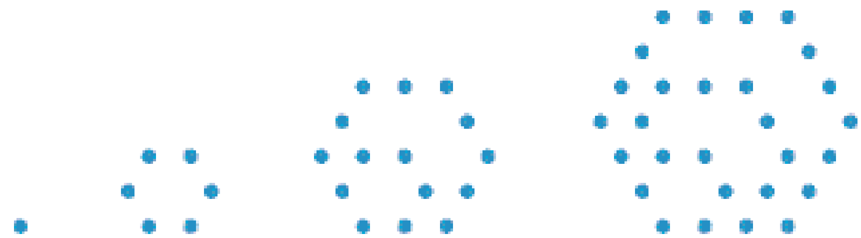
CHALLENGE yourself

Make a quadratic sequence for each given property.

(a) The first term is 3.

(b) Every term is a negative integer.

12. The figures representing the first four *hexagonal* numbers are shown.

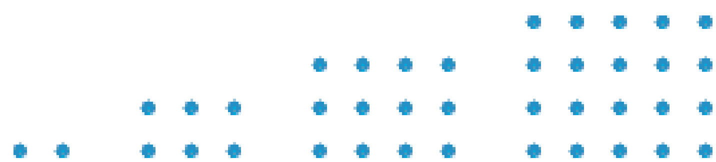


(a) Count the number of dots in each hexagonal figure. Use these hexagonal numbers to create a sequence of four terms. Use a pattern to find the next term in the sequence.

(b) Use the results of Question 11 to predict D_2. Check your prediction.

(c) Identify the type of sequence created in part (a).

13. The figures representing the first four *oblong* numbers are shown.



(a) Count the number of dots in each oblong figure. Use these oblong numbers to create a sequence of four terms. Use a pattern to find the next two terms in the sequence.

(b) Identify the type of sequence created in part (a).

14. Identify the arithmetic and quadratic sequences.

(a) {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, …}

(b) {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …}

(c) {4, 7, 12, 19, 28, 39, 52, 67, 84, …}

(d) $\left\{\frac{3}{4}, 1\frac{1}{2}, 2\frac{1}{4}, 3, 3\frac{3}{4}, 4\frac{1}{2}, 5\frac{1}{4}, 6, 6\frac{3}{4}, 7\frac{1}{2}, 8\frac{1}{4}, 9, \ldots\right\}$

(e) {1, 8, 27, 64, 125, 216, 343, 512, 729, 1000 …}

(f) {–2, –10, –24, –44, –70, –102, –140, –184, –234, –290 …}

15. A truck is travelling toward a bridge at 20 m/s and is 100 m away from passing under the bridge.

(a) How far away from the bridge is the truck after 1 s? After 2 s? After 3 s?

(b) What type of sequence is formed using the distance in metres from the truck to the bridge each second? Explain.

(c) When does the truck pass under the bridge?

16. For a movie stunt, a stuntwoman jumps off a building to land in a safety net. After she jumps, her height above the ground in metres after each tenth of a second is given by the sequence {12.951, 12.804, 12.559, 12.216, 11.775, …}.

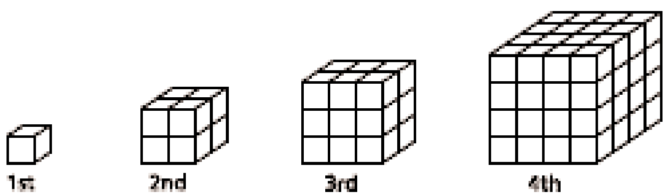
(a) Use sequences of differences to determine the type of sequence it is.

(b) What is the next term in the sequence for her height after each tenth of a second?

Investigation 6

Cubic Number Patterns

In Investigation 5, you found that sequences of triangular and square numbers form quadratic sequences. In this Investigation, you will explore the properties of sequences of *cubic* numbers.



Think about...

The Diagrams

Why is the number of smaller cubes in each larger cube called a *cubic* number?

Purpose

Explore cubic sequences and their sequences of differences.

Procedure

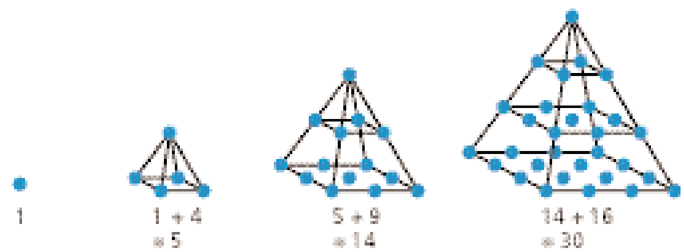
A. Count the number of small cubes in the first, second, third, and fourth cubes. Use these numbers to form the terms of a sequence.

B. Use a pattern or a rule to extend the sequence to 10 terms.

C. Find the sequences of differences D_1, D_2, and D_3 for the sequence in Step B. Which sequence of differences has a constant term? What is the significance of this? Is the sequence in Step B linear, quadratic, or some other type of sequence?

Investigation Questions

17. These diagrams represent cannon balls stacked in the shape of a square pyramid. The number of balls in the stacks can be written as the sequence {1, 5, 14, 30}.



(a) Make a sequence with seven terms by finding the number of cannon balls in square pyramids with one to seven layers.

(b) Find the sequences of differences D_1, D_2, and D_3 for the sequence in part (a). Which sequence of differences has a constant term?

(c) Based on the sequences of differences of cubic numbers that you found in part (b), what type of relation or rule is likely needed to generate the cannon ball sequence?

18. When D_1 is constant, the sequence is an arithmetic sequence. When D_2 is constant, the sequence is a quadratic sequence involving squares of numbers. What type of sequence do you think exists when D_3 is a constant number?

19. The relation $t_n = \dfrac{2n^3 + 3n^2 + n}{6}$ can be used to find the total number of cannon balls in a square pyramid with n levels.

(a) Why do you think the relation is called a *cubic* relation?

(b) Show that this relation builds the sequence that you made in Question 17(a).

(c) Use the relation to find the number of cannon balls in a square pyramid with 50 layers.

20. If apples in a grocery store are stacked in the shape of a triangular pyramid, then one layer contains 1 apple, two layers contain 4 apples, three layers contain 10 apples, and four layers contain 20 apples. The number of apples in the stacks can be written as the sequence {1, 4, 10, 20}.

Think about...

Question 17

What do you notice about the shape of each layer of cannon balls?

How many cannon balls will be in the bottom layer of the fifth stack of cannon balls? What is the fifth term in the sequence?

Explain how you know that the sequence is neither an arithmetic sequence nor a quadratic sequence.

Did You Know?

In the early days of artillery, cannon balls were stacked in pyramids. As a result, the total number of cannon balls in a pyramid could be found by counting the number of layers of the pyramid.

CHALLENGE yourself

In 1994, 10 people built a square pyramid of 4900 cans. Find the number of layers in the pyramid.

(a) Compare the number of apples in the bottom layers with the sequence of triangular numbers that you created in Question 9(a) of Investigation 5. Explain how the sequence of triangular numbers can help you to find

(i) the number of apples in the bottom layer of the next stack

(ii) the total number of apples in the next stack

(b) Create a sequence with seven terms by finding the number of apples in triangular pyramids with one to seven layers.

(c) Find the sequences of differences D_1, D_2, and D_3 for the sequence in part (b). Which sequence of differences has a constant term?

(d) Is the sequence in part (b) arithmetic, quadratic, or cubic?

21. A balloon manufacturer makes large spherical balloons filled with helium for displays at special events. The table below shows the number of kilolitres of helium needed to fill balloons with various circumferences.

Circumference (m)	10	20	30	40	50	60	70
Helium (kL)	16.89	135.09	455.95	1080.76	2110.86	3647.56	5792.19

(a) Form a sequence of seven terms using the second row in the table.

(b) Find the sequences of differences D_1, D_2, and D_3 for the sequence in part (a). Which sequence of differences has a constant term?

(c) Is the sequence in part (a) arithmetic, quadratic, or cubic?

(d) What type of relationship exists between the circumference and the amount of helium needed for a spherical balloon?

Check Your Understanding

22. (a) Each quadratic sequence and cubic sequence in Section 1.3 is called a *power sequence*. Write your own definition of a power sequence.

(b) What is the relationship between a sequence of differences, D_n, and the degree or greatest exponent of the relation generating a power sequence?

23. Generate a quadratic or cubic sequence containing at least 10 terms. Exchange sequences with another student. Use sequences of differences to find the type of sequence used by the other student.

Think about...

Sequences Generated by Cubic Relations

What seems to be the relationship between the degree or greatest exponent of a cubic relation and D_3?

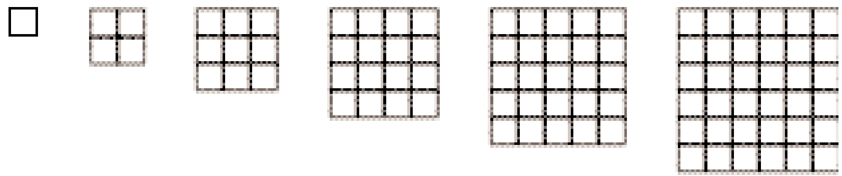
CHALLENGE yourself

Create and name each type of sequence after each operation is performed.

(a) Multiply the corresponding terms of two arithmetic sequences.

(b) Multiply the corresponding terms of an arithmetic sequence and a quadratic sequence.

24. The total number of squares in a 1-by-1 grid is 1 and in a 2-by-2 grid is 5. In the 2-by-2 grid, there are four 1-by-1 squares and one 2-by-2 square, or 5 squares. These results can be written as the sequence {1, 5, ...}.



(a) Find the number of squares in a 3-by-3 grid, a 4-by-4 grid, a 5-by-5 grid, and a 6-by-6 grid. Verify your numbers with others in the class.

(b) Use the results in part (a) to continue the sequence to six terms.

(c) Use sequences of differences to find whether the sequence is arithmetic, quadratic, or cubic.

(d) Describe and use a pattern to find the number of squares on an 8-by-8 chessboard.

25. In Question 13, you made a sequence by counting the number of dots in oblong figures.

(a) Make another sequence with at least six terms by finding the *total* number of dots as each oblong figure is added to the other. The first three terms of the sequence are {2, 2 + 6 = 8, 2 + 6 + 12 = 20} or {2, 8, 20}.

(b) What type of sequence is it? Explain.

(c) Use a pattern to find the total number of dots in the first seven oblong numbers. Describe the pattern.

26. In Investigation 6, you created a sequence by counting the number of small cubes in 1-by-1-by-1 cubes, 2-by-2-by-2 cubes, 3-by-3-by-3 cubes, and so forth.

(a) Create another sequence with at least six terms by finding the *total* number of small cubes as each larger cube is added to the other. The first three terms of the sequence are {1, 1 + 8 = 9, 1 + 8 + 27 = 36}, or {1, 9, 36}.

(b) Use sequences of differences to find whether the sequence is arithmetic, quadratic, cubic, or another kind of sequence. If it is another kind of sequence, what kind do you think it might be?

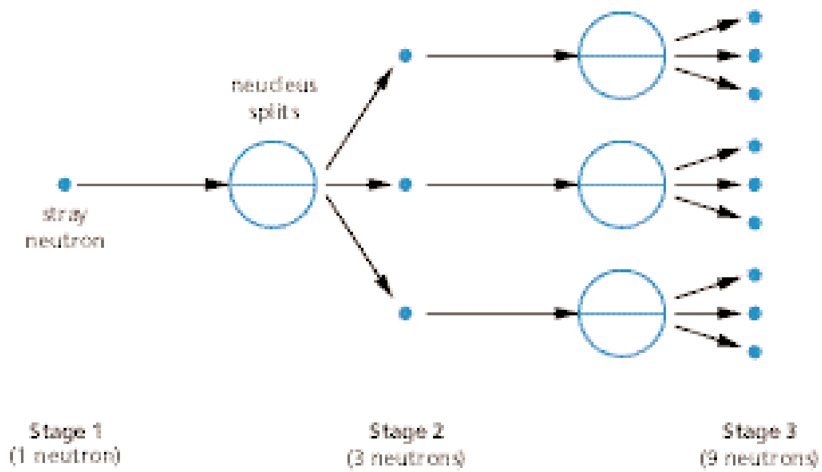
(c) Use a pattern to find the total number of small cubes in the first seven larger cubes.

Investigation 7

Patterns of Rapid Growth

Every atom has a central core called the nucleus. The nucleus of a uranium atom can be split into smaller parts if it is struck by a stray particle called a neutron.

The diagram represents a nuclear chain reaction. A stray neutron splits a uranium nucleus and releases three other neutrons, each capable of splitting another uranium nucleus. The repeated splitting of nuclei and releasing of neutrons results in the tremendous energy output of nuclear reactors and bombs.



The number of stray neutrons at each of the first three stages can be written as the sequence {1, 3, 9}. How many stray neutrons will there be after 10 stages of this chain reaction?

Purpose

Explore the properties of number patterns that grow very rapidly.

Procedure

A. Draw a diagram for the number of stray neutrons on the next split.

B. Make a sequence with at least 10 terms to show the number of stray neutrons at each stage in this chain reaction.

C. Graph the number of stray neutrons at each stage versus the stage number. Describe the shape of your graph.

D. Use the sequence in Step B to make sequences of differences D_1, D_2, and D_3. What number patterns do you see?

Did You Know?

Use the Internet to learn more about nuclear energy and nuclear chain reactions. The Web site **www.geocities.com/Athens/Olympus/5297/** contains an animated illustration of the splitting of an atom into several particles.

E. Is the sequence in Step B an arithmetic sequence, a quadratic sequence, a cubic sequence, or another type of sequence? Explain.

F. Refer to the sequence in Step B and D_1, D_2, and D_3 in Step D. How can you generate the terms in the sequence?

G. Create another sequence by dividing each term in the sequence in Step B by the previous term. Describe what you notice about each term in this new sequence.

Did You Know?

In 2000, several people died in Walkerton, Ontario, after drinking water contaminated by *E. coli* bacteria. During the same year, many people in the Atlantic provinces became ill after eating hamburger meat containing *E. coli.*

Investigation Questions

27. Suppose five neutrons existed at the beginning of the nuclear chain reaction instead of one neutron.

(a) Complete Steps B to G using this information.

(b) How many stray neutrons will there be after 10 stages in this chain reaction?

28. The first term in a sequence is 4. Each succeeding term is created by multiplying the previous term by 5.

(a) Write a sequence of 10 terms using this information.

(b) Is the sequence in part (a) an arithmetic sequence, a quadratic sequence, a cubic sequence, or a **geometric sequence**? Explain.

geometric sequence—a number sequence that is built by always multiplying by the same number; for example, the sequence {2, 6, 18, 54, ...} is built by multiplying the previous term by 3

29. Pick any number as the first term in a sequence. Pick a second number and create a sequence of 10 terms by repeatedly multiplying each previous term by your second number. Exchange your sequence with another student.

Decide whether the sequence that you are given is an arithmetic sequence, a quadratic sequence, a cubic sequence, or a geometric sequence.

Check Your Understanding

30. Michael Crichton, author of *Jurassic Park*, wrote the following paragraph in his book *The Andromeda Strain* (Dell, 1969):

> *The mathematics of uncontrolled growth are frightening. A single cell of the bacterium* E. coli *would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of* E. coli *could produce a super-colony equal in size and weight to the entire planet Earth.*