1.3 Connecting the Region and the Solution

Graphing the feasible region is a method of identifying all possible solutions to a problem. In this section, you will work with a procedure that allows you to find the **optimal solution** to a problem more directly and exactly.

optimal solution—the solution that best meets the constraints in the problem and allows you to maximize or minimize a specific quantity

Investigation 4

Finding the Best Solution

Spinney Manufacturing decides to branch out into different markets. They plan to make baseball hats and visors to promote to these new markets. They will make a cutting machine available for 2 h (120 min) and a sewing machine available for 1 h (60 min) each day. Each baseball hat takes 4 min on the cutting machine and 3 min on the sewing machine. Each visor takes 3 min on the cutting machine and 1 min on the sewing machine.

- How many of each item can they make each day? Give three possible answers.
- If the profit on a baseball hat is \$1.10 and the profit on a visor is \$0.60, estimate the maximum profit they can make.

A. Copy and complete the table below to organize the information.

	Time on cutting machine	Time on sewing machine	Profit
Hats (h)			
Visors (v)			

B. Given that $h \geq 0$ and $v \geq 0$, use the information in the table to write two more inequalities to represent this situation.

C. Graph the inequalities to identify the feasible region.

D. The profit equation can be written as $P = 1.1h + 0.6v$. Explain why this equation represents the profit.

E. Choose six points in the feasible region. Calculate the profit for each of the points you chose. Organize a table like the one in the margin.

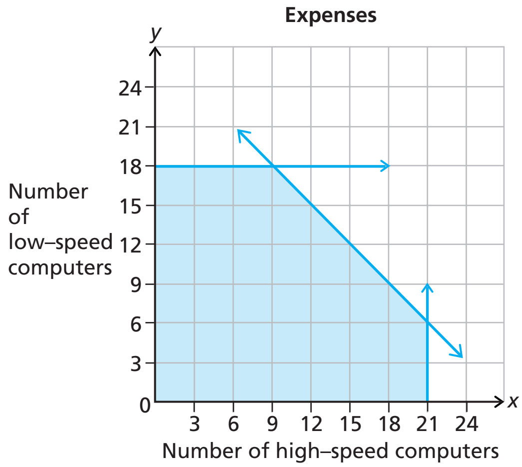
Number of hats	Number of visors	Profit

F. Compare your points, and the profit you calculated, with others in the class.

G. Where does it appear that the maximum profit will be found?

Investigation Questions

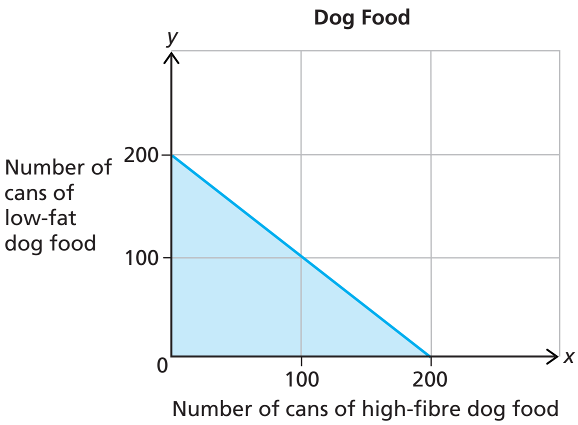
1. The expenses for a computer manufacturer are given by $E = 3x + 4y$, where x is the number of high-speed computers manufactured and y is the number of low-speed computers manufactured. The graph represents the manufacturing constraints.



(a) What is the maximum expense?

(b) What is the minimum expense?

2. A dog food manufacturer sells a high-fibre dog food and a low-fat dog food. The profit for the company can be represented by $P = 3x + 2.50y$, where x represents the number of cans of high-fibre dog food, and y represents the number of cans of low-fat dog food. The graph below represents the manufacturing constraints.



(a) What is the minimum profit?

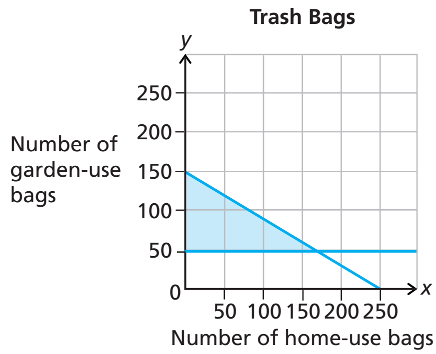
(b) What is the maximum profit?

Think about...

Questions 1 to 3

What is the relationship between the vertices of the feasible region and the maximum and minimum values?

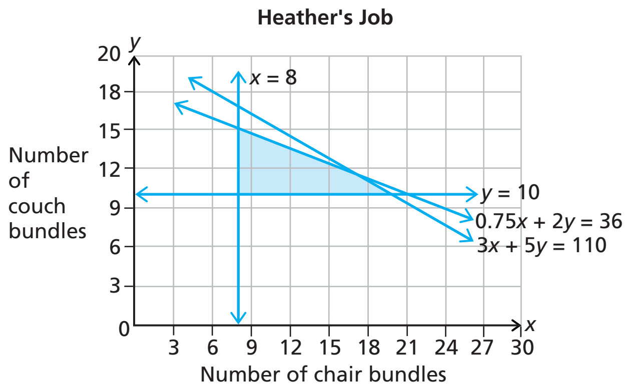
3. A trash bag manufacturer sells trash bags for home use and garden use. The profit for the company can be represented by $P = 2.5x + 0.75y$, where x represents the number of trash bags for home use, and y represents the number of bags for garden use. The graph represents the constraints.



   (a) What are the coordinates of each vertex?

   (b) What is the maximum profit? the minimum profit?

4. At the end of Section 1.2, you finished graphing the constraints for Heather's job. The graph looks like the one shown below. You could use Blackline Master 1.1.1 to help you with the questions below.



   (a) Predict the place on the graph that you think would represent Heather's maximum income; Heather's minimum income.

   (b) Explain why you chose the points you did.

   (c) Use the information given in Heather's problem.

      (i) Write an income equation for Heather's job.

      (ii) Calculate her income for the point you chose.

      (iii) Is this the only point in the feasible region where she would make this income?

   (d) Use the scale in the margin to describe your confidence in your solution. Explain your choice.

   (e) Is there another method for finding the solution that could increase your confidence in your solution? Explain.

Confidence Scale

- 0— no confidence
- 3— some confidence, but have a feeling the solution is not accurate
- 5— fairly confident
- 8— quite confident, but not 100% positive that there is not a better solution
- 10—completely confident with no doubt that the optimal solution has been found

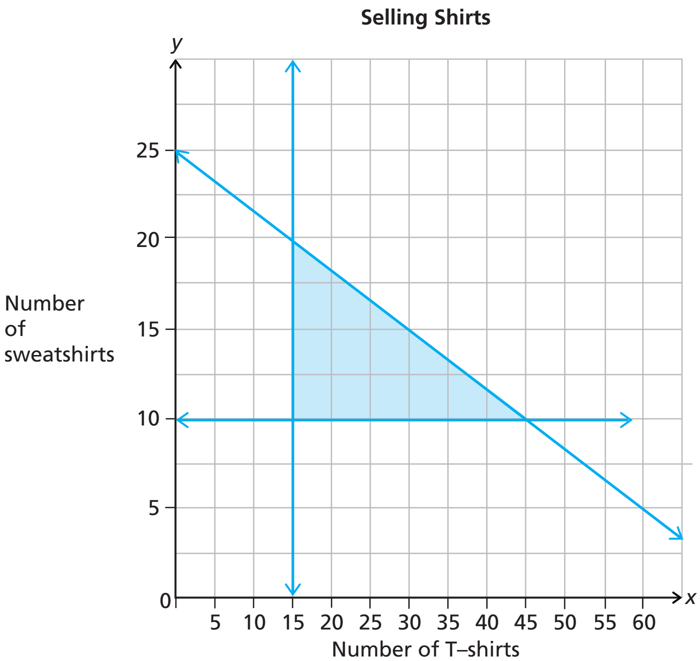
5. The process you have used to find Heather's optimal income is called **linear programming**. In your own words, describe the process of linear programming. Remember to include:
   - how to find the feasible region
   - how to find the income function
   - how the income function is used in the feasible region to find the maximum income

   Use an example of your own to help with your description.

linear programming—a process that uses a number of linear inequalities, representing constraints, where the objective is to maximize or minimize a function

Check Your Understanding

6. (a) Sheila is making T-shirts and sweatshirts to sell. The graph shows her constraints. She earns \$8.00 for every T-shirt and \$16.00 for every sweatshirt. How many of each must Sheila sell to earn the maximum income?

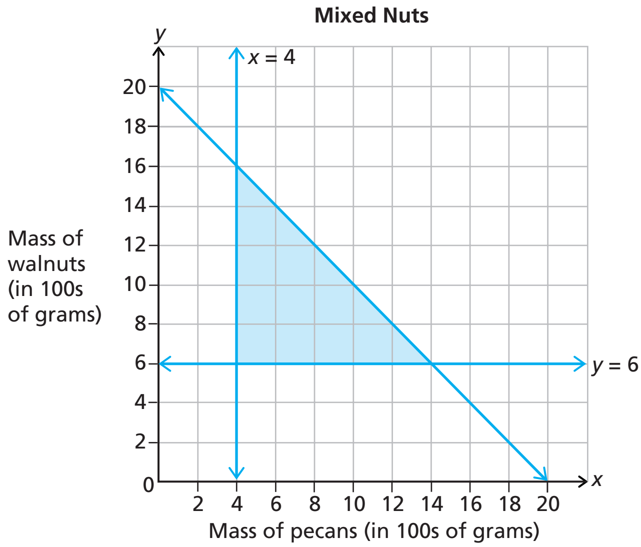


Think about...

Question 6

For each part, explain what each inequality shown on the graphs could represent.

(b) Pecans cost \$3/100 g and walnuts cost \$1.50/100 g. Anna wants to make a bag of mixed nuts. The graph shows her constraints. What is the minimum cost of a mixed bag of the nuts?



— Note —

Like a maximum, a minimum will also occur at an intersection point of the border of the feasible region.

FOCUS E Solving a System of Equations by Estimating the Vertices

Finding the intersection points of the boundary lines helped you to find Heather's maximum income. The following example shows how to find the coordinates of the intersection point of two graphs.

Using two equations at the same time to find an intersection point is called solving a **system of equations**.

system of equations—a set of equations used to describe a situation for which there is often a common solution

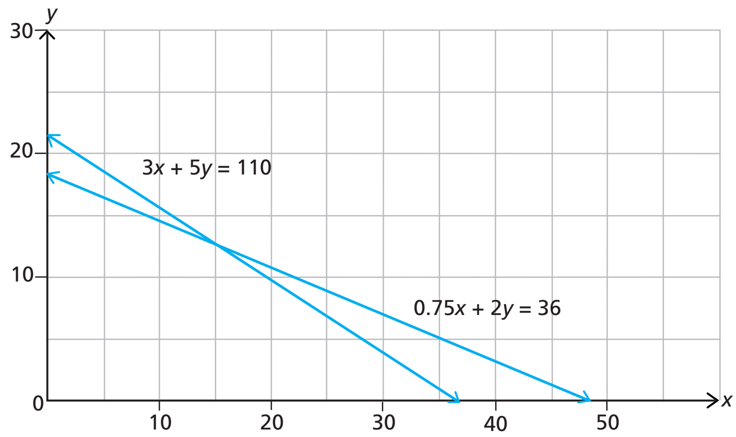
Example 1

To find one of the intersection points of her region, Heather needed to graph $0.75x + 2y = 36$ and $3x + 5y = 110$. Estimate the point of intersection.

Solution

Method 1 *Graphing*

- Solve both equations for y so they can easily be graphed.
- Plot the graphs of both lines on the same coordinate axes.
- Estimate the point where the two lines intersect.



On the graph, it looks like the point of intersection is (18, 12). Finding this point is called **solving a system of equations**.

Sometimes, the interesection points are not whole numbers and it becomes difficult to estimate the exact intersection point. It can be useful to use other methods to find these points.

– Note –

The equation $3x + 5y = 110$ can be rearranged to solve for y as follows:

$$3x + 5y = 110$$
$$3x - 3x + 5y = 110 - 3x$$
$$5y = 110 - 3x$$
$$\frac{5y}{5} = \frac{110}{5} - \frac{3x}{5}$$
$$y = 22 - 0.6x$$
$$y = -0.6x + 22$$

- Solve for y in $0.75x + 2y = 36$.
- How could these equations be used to estimate the point of intersection using graphing technology?

solving a system of equations—finding the values of the variables that satisfy both equations in a system of equations

Method 2 ***Technology***
You can also use technology to estimate the point of intersection.

Graph the two equations on the same coordinate grid. Use the trace function and trace along one of the graphs. Stop when you get to the point of intersection. What is the point of intersection? Is this an exact value? Why?

– Note –

You can use the intersection feature to find a point of intersection.

- Enter the two equations.
- Press CALC:5.
- Show the lines on the grid.
- Press ENTER until "Intersection point" appears on the screen.

Method 3 ***Using a Table of Values***
You can use the equations in the form $y = mx + b$ and the table feature of a graphing calculator (if available) to locate the point of intersection.

- Enter the equations $0.75x + 2y = 36$ and $3x + 5y = 110$.
- Investigate the tables of values. Is there a common x- and y-value?
- Discuss this with others in your class. You may be able to complete the tables of values without technology.

Focus Questions

7. (a) Solve this system of equations by graphing.

$y = 6x - 1$ $\qquad$ $y = x - 8$

(b) Do you think the intersection point can be found accurately using this method? Why or why not?

(c) Solve the system of equations using graphing technology. Does this method provide an accurate solution? Explain.

– Note –

In Question 7, make sure you draw the graphs before you do part (b).

8. (a) Graph two lines that will never intersect. Write their equations. They represent a system of equations with no solution. What do you notice?
   (b) Exchange your graphs with a classmate. Will their graphs never intersect? How can you tell?
   (c) What do you know about the slopes of these lines?

Check Your Understanding

9. Solve each system of equations.

   (a) $y = 2x - 1$, $y = 4x - 3$
   (b) $y = 3x + 1$, $y = 2x - 1$
   (c) $y = x + 4$, $y = 4x - 2$
   (d) $y = -4x + 5$, $y = x - 5$
   (e) $y = x - 5$, $y = 3x - 4$
   (f) $y = 3x + 4$, $y = 7 - x$

10. Solve each system of equations.

   (a) $2x + y + 2 = 0$, $3x + y + 13 = 0$
   (b) $2x - y = -5$, $x + y - 2 = 0$
   (c) $2x = y + 8$, $x + 3y - 4 = 0$

11. A ship is sailing on a path given by the equation $y = 4x + 6$. A weather balloon is flying on a path given by the equation $y = 4 - 2x$. Find the coordinates at which the weather balloon flies over the ship.

FOCUS F Solving Systems of Equations Algebraically

Think about...

Focus F

This method of solving a system of equations is often called "the comparison method." Why do you think this name is used?

The graphing method Heather used to find the point of intersection in Focus E provided only an estimate. Because she was investigating the maximum amount of money she could earn, she wanted a more exact method of finding the intersection point. She chose to solve the system of equations using the following method.

In Focus E, Heather rewrote the equations in the form $y = mx + b$ so that she could graph the equations. This system of equations can be solved as shown.

$y = 22 - 0.6x$ ①
$y = 18 - 0.375x$ ②

Since Heather is finding the point of intersection, she knows that the y-values are equal.

$22 - 0.6x = 18 - 0.375x$

Heather then solved this linear equation.

$$\begin{aligned} 22 - 0.6x + 0.6x &= 18 - 0.375x + 0.6x \\ 22 &= 18 + 0.225x \\ 4 &= 0.225x \\ \frac{4}{0.225} &= \frac{0.225x}{0.225} \\ x &= 17.78 \end{aligned}$$

The value of y can be found by substituting into either equation.

$$\begin{aligned} y &= 22 - 0.6x \quad ① \\ &= 22 - 0.6(17.78) \\ &= 11.33 \end{aligned}$$

The point of intersection (17.78, 11.33) seems reasonable to Heather because she estimated the point to be (18, 12) by reading her graph.

– Note –

You should verify the point of intersection using the equation you did not use to find the value of y.

Focus Questions

12. Solve Question 11 using the algebraic approach.

13. List the advantages and disadvantages of solving a system of equations by:

(a) graphing the equations

(b) using the comparison method

(c) using a table of values

(d) Which method do you prefer to use to find the intersection point of two linear equations? Give reasons for your answer and use an example of your own to support your answer.

14. If you found the intersection points of the borders of the feasible region in Heather's problem using algebra, would you still need the graph to solve the problem? Explain.

Check Your Understanding

15. Solve. Verify your solutions using graphing technology.

(a) $y = x - 4$
$y = 4x + 5$

(b) $y = x + 1$
$y = 3x - 5$

(c) $y = 2x - 1$
$y = x + 7$

(d) $y = 7 - 3x$
$y = 2x - 3$

16. Marion usually rents a car when she travels. One company charges $50 plus 25¢/km. The other charges $90 plus 20¢/km. Find the number of kilometres travelled when the costs are equal.

17. Solve these systems of equations.

(a) $2x + y = 5$
$4x - y = 9$

(b) $3x - y = 5$
$y - 2x = 2$

(c) $4x - y = 9$
$x - y = 3$

18. Jane makes macramé jewellery. The number of bracelets she makes is represented by b, and the number of necklaces she makes is represented by n. Find the number of necklaces and the number of bracelets made each week in each of the following systems. Are there any answers that are unreasonable? Explain.

(a) $b + 2n = 3$
$b - n = 3$

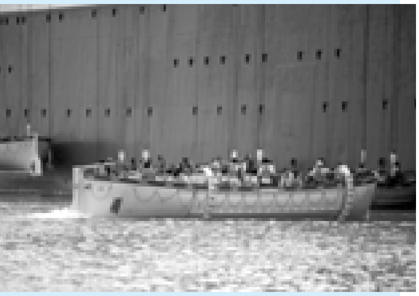
(b) $2b + n = 6$
$4b - n = 9$

(c) $2b + 4n = 5$
$b + n = 2$

(d) $3b + n = 12$
$2b + 3n = 30$

(e) $b = 5n + 8$
$b = 2n - 1$

(f) $b - 2n = 3$
$b + n = 6$



19. A lifeboat is drifting with survivors. It gets caught in a current and starts travelling along a path given by $2p - 3q = 12$. A rescue ship is travelling as quickly as possible to pick up the survivors. It is travelling a path given by $3p - 2q = 13$.

(a) Find the coordinates of the point at which the rescue ship will meet the survivors.

(b) Suppose the rescue ship started at a port located at (0, 0). How far will the rescue ship need to travel to meet the survivors?

20. In a consulting firm, the senior partner bills her time at a rate of \$125/h and the junior partner at a rate of \$75/h. The cost and time to complete a job for one of their clients are quoted at \$10 000 and 100 h. Find the amount of time each partner should work to complete this job.

FOCUS G Solving Systems of Equations With Fractions

Some equations you are solving may contain fractions. In these cases, you should multiply the entire equation by a factor that converts the fraction into a whole number. This will allow you to solve the system of equations using the methods that you have already learned.

Example 2

Solve the following system of equations.

$\frac{1}{2}x - 3y = 3$

$\frac{2}{3}x + 4y = -2$

Solution

$\frac{1}{2}x - 3y = 3$ ①

$\frac{2}{3}x + 4y = -2$ ②

Multiply equation ① by 2, which is the common denominator of all fractions in the equation.

$2(\frac{1}{2}x - 3y = 3)$ ①

$x - 6y = 6$ ③

Multiply equation ② by 3, which is the common denominator of all fractions in the equation.

$3(\frac{2}{3}x + 4y = -2)$ ②

$2x + 12y = -6$ ④

The resulting system of equations is:

$x - 6y = 6$ ③

$2x + 12y = -6$ ④

From equation ③, $x = 6 + 6y$

From equation ④, $x = -3 - 6y$

Since you are finding the point of intersection, the x-values must be equal, therefore

$$\begin{aligned} 6 + 6y &= -3 - 6y \\ 9 &= -12y \\ -0.75 &= y \end{aligned}$$

$$\begin{aligned} x &= 6 + 6y \\ &= 6 + 6(-0.75) \\ &= 1.5 \end{aligned}$$

The point of intersection is (1.5, –0.75).

–Note–
In equation 4, you can divide each term by 2 to simplify the equation. Why do you think equations 3 and 4 are in the form of $x = \ldots$?

Focus Questions

21. Solve each system of equations.

(a) $\frac{1}{3}x - y = -2$
$x - \frac{2}{3}y = 1$

(b) $y + \frac{3}{4}x = 4$
$x - \frac{1}{4}y = -1$

(c) $\frac{1}{2}x + 2y = 4$
$x - y = 2$

(d) $2x - \frac{1}{2}y = 4$
$\frac{2}{3}x = y - 2$

(e) $\frac{1}{2}x + \frac{1}{3}y = 5$
$\frac{1}{3}y - x = -4$

(f) $\frac{1}{2}a + 3b = 2$
$\frac{2}{3}a + b = 5$

22. In each of the following systems of equations, x represents the number of video games played per hour and y represents the number of pinball games played per hour at a local video arcade. Find the number of video games and the number of pinball games played per hour in each situation. Are the answers reasonable? Explain.

(a) $1.9x - 3.3 = 2.8y$ and $4.1y + 5.2x - 8.3 = 0$

(b) $2.4x = 2.2y - 3.2$ and $1.6x + 1.2y = 3.2$

(c) $3(x - 1) - 2(y + 1) = -14$ and $3x - y = -6$

– Note –

Throughout this chapter, you have learned to use the following steps to solve a linear programming problem:
(i) List the constraints.
(ii) Graph the feasible region.
(iii) Locate the point of intersection.
(iv) Calculate the optimal solution.

23. (a) Sketch the graph of the region defined by each pair of inequalities. Find the maximum profit if the profit function is given by $P = 3x + 2y$. For each inequality, the values of x and y are greater than zero.

(i) $y \le 4 - 2x$ and $y \ge 1$

(ii) $2y > x + 12$ and $3x - 5 < y$

(iii) $2x + 3y \le 6$ and $3x + 2y \ge 6$

(b) Find the profit at each vertex. What is the maximum profit?

FOCUS H

Heather's Maximum Income

Heather wants to know how she can make the most money working at Spinney Manufacturing. She has the following information.

- Heather cuts fabric for couches and chairs at Spinney Manufacturing.
- She is paid a flat fee for each bundle of fabric that she cuts.
- A bundle includes all of the pieces of fabric that are sewn together to make a chair or couch.
- Heather is paid $12.00 for each couch bundle that she cuts.
- She is paid $4.25 for each chair bundle that she cuts.
- If she completes only a part of a bundle, she is paid for only that part. For example, she is paid $6.00 if she cuts half of a couch bundle.
- She can work a maximum of 36 h every two weeks.
- It takes her 45 min to cut fabric for a chair.
- It takes her 2 h to cut fabric for a couch.
- At least eight chair bundles must be cut every two weeks.
- At least ten chair bundles must be cut every two weeks.
- She may use a maximum of 110 m of fabric in a two-week period.
- A chair bundle uses 3 m of fabric.
- A couch bundle uses 5 m of fabric.

Procedure

A.
- List the constraints on Heather's income.
- Describe how you wrote these constraints as inequalities.
- Explain how you plotted the graph of the feasible region.
- How did you estimate her maximum income using the graph?

B. The exact value of the intersection point can be found by solving systems of equations algebraically.

- Write the equations of the boundary lines.
- Find the point where the lines intersect to give the point where the maximum income was found.

Check Your Understanding

Solve at least two of Questions 24 to 28 using the method of linear programming used to solve Heather's problem.

24. A firm manufactures bicycles and tricycles. The company makes a profit of $50 on each bicycle and $30 on each tricycle. Find the maximum profit that can be made under the following conditions.

- A maximum of 80 frames can be made each month.
- It takes 2 h to assemble a bicycle.
- It takes 1 h to assemble a tricycle.
- The assembler is available 100 h each month.

25. Carpet City manufactures wool and nylon fibre rugs. The warehouse has 915 lots of wool fibre and 1120 lots of nylon fibre.

- A Wearwell rug uses 15 lots of wool fibre and 35 lots of nylon fibre.
- An Ultraguard rug uses 40 lots of wool fibre and 45 lots of nylon fibre.
- It takes 4 h to manufacture a Wearwell rug.
- It takes 3 h to manufacture an Ultraguard rug.
- The profit on a Wearwell rug is $300.
- The profit on an Ultraguard rug is $180.
- There are 98 h of manufacturing time available each week.

How many of each type of carpet must be manufactured to maximize profits?

26. A sports equipment manufacturer makes basketballs and soccer balls. The time required on each machine is shown.

	Time on Machine A	Time on Machine B
Basketball	2 min	1 min
Soccer ball	1 min	2 min

- Machine A is available for 110 min per day.
- Machine B is available for 140 min per day.
- The profit on a basketball is $2.30.
- The profit on a soccer ball is $1.95.

Calculate the number of basketballs and soccer balls that should be manufactured to make the maximum profit.

Think about...

Each Solution

Where does the optimal solution (the maximum or minimum) of any linear programming problem appear to occur?

27. To produce top quality apples, Dan needs to use at least:

- 7.3 kg of Nutrient A per apple tree
- 4.7 kg of Nutrient B per apple tree

The amounts of A and B in the brands are shown below.

Supplier	Amount of Nutrient A	Amount of Nutrient B	Cost per kilogram
Erunam	40%	60%	$2.40
Goodwin	90%	10%	$3.00

Dan needs to create a fertilizer that provides the required amount of Nutrients A and B at the least cost. How many kilograms of each brand should he use?

28. Universe Corporation makes two types of sports shoes: an aerobics shoe and a high-top basketball shoe. The shoes are assembled by a machine and then finished by hand.

- Each aerobics shoe requires 0.25 h on the machine and 0.1 h by hand.
- Each basketball shoe requires 0.15 h on the machine and 0.2 h by hand.
- The machine cannot be used more than 900 h per month.
- The people who can work by hand will work no more than 500 h per month.
- There is an $18 profit on the sale of each aerobics shoe. There is a $10 profit on the sale of each basketball shoe.

How many of each type of shoe should be made to maximize profit?

PUTTING IT TOGETHER

PROBLEM SOLVING

As the sales manager for CTPN radio station, you want to sell exclusive advertising rights on a half-hour music program called "Martina's Magic." Music will be played for x min, commercials will run for y min, and Martina will talk the rest of the time.

You want to find the maximum number of listeners you can expect to have at any time throughout the show because potential advertising customers will need this information. Through research, you determine the following information:

- The audience increases by 500 people for every minute that music is played.
- The audience decreases by 500 people for every minute that Martina speaks.
- To be profitable, the program must run at least 5 min of commercials.
- Government regulations state that a maximum of 12 min of commercial time be used per half hour.
- The amount of time that commercials are played must not exceed the amount of time that music is played.

(a) Write inequalities to represent the constraints. Let x represent the length of a song and y represent the length of a commercial.

(b) Graph the region represented by the inequalities in part (a).

(c) Write an equation for the total number of listeners as $N = ...$

(d) What is the maximum number of listeners for the show?

EXTENSION

Another important consideration is the cost to produce the show. The sales manager provides the following costs:

- Royalty payments of $400 per minute are required.
- Commercials cost $200 per minute to make and run.
- Each minute that Martina speaks costs $300.

(a) Write an equation in the form $C = \boxed{}$ that can be used to calculate the cost to produce the show.
(b) What is the minimum cost for the show?
(c) What is the minimum cost to attract the maximum number of listeners?

CASE STUDY: THE DIET PROBLEM

www.mcs.anl.gov/home/otc/Guide/CaseStudies/diet

– *Note* –
These results are for information only. The optimal diet may not be recommended for you by your doctor.

The development of the diet problem was motivated by the American army's desire to meet the nutritional requirements of soldiers at a minimal cost. The initial solution was found in 1939 by nine people working 120 d each. The problem is discussed at the Web site shown above.

The goal of "The Diet Problem" is to find the cheapest combination of foods that will satisfy all of the daily nutritional requirements of a person. The objective is to minimize cost and meet constraints that require that nutritional needs be satisfied. You must include constraints that regulate the number of calories and amounts of vitamins, minerals, fats, sodium, and cholesterol in the diet.

When doing the problem:

- Choose your own foods for your menu.
- Try to create an optimally low-cost menu that meets the nutritional constraints.

This problem is presented on the Internet using built-in computer programs that allow you to solve the problem using constraints that you impose. Try it! You may be interested to discover how people select a nutritionally balanced diet. Remember, however, that what is mathematically acceptable may not be practical, possible, or real.

Have fun with this Case Study!

REVIEW

Key Terms

	page
constraint	14
domain	3
feasible region	18
intersection point	26
inequality	6
linear programming	25
optimal solution	22
solving a system of equations	27
system of equations	26

You Will Be Expected To

- Identify the variables and constants in a problem.
- Graph linear inequalities in two variables to define a feasible region.
- Identify a quantity to be maximized or minimized, and express the quantity in terms of the variables.
- Solve a system of linear equations.
- Formalize the relationship between the feasible region, the intersection points on the border of the feasible region, and the optimal solution to a problem.

Summary of Key Concepts

This problem connects the concepts covered in this chapter.

Vince is the company sales representative for Newfoundland and New Brunswick. Since he must pay his own expenses, he reviews them from the last few years to develop a budget.

- He decides upon an "average" travel budget of $100 per day when travelling in Newfoundland, and an "average" daily travel budget of $120 when travelling in New Brunswick. To be profitable, he needs to stay within an annual travel budget of $18 000.
- As part of his contract, he must spend at least 50 d in Newfoundland and 60 d in New Brunswick.
- Vince has leased a car. In Newfoundland, he travels an average of 275 km per day. In New Brunswick, he travels an average of 110 km per day. His lease lets him travel a maximum of 26 400 km per year with no additional charges.
- Historically, sales average $3000 per day in Newfoundland and $2800 per day in New Brunswick.

What is the maximum sales figure Vince can expect? How many days should he travel in each province?

1.1 Exploring an Optimization Problem

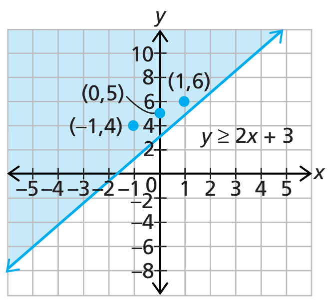
In Section 1.1, some necessary skills needed to solve this problem were reviewed and explored. In the summary for Sections 1.2 and 1.3, the solution to the problem will be shown.

Example 1

Find the region that represents $y \geq 2x + 3$ and shade the graph.

Solution

(0 , 5), (1 , 6), and (–1 , 4) are points that satisfy the inequality. Thus, they lie in the shaded region.



1.2 Exploring Possible Solutions

Example 2

(a) Write inequalities to represent Vince's problem.

(b) Graph these inequalities and shade the feasible region.

(c) List some possible solutions in the feasible region.

Solution

(a) For this problem, let p represent the number of days travelling in New Brunswick, and let n represent the number of days travelling in Newfoundland.

The constraints can now be classified under these three headings.

Constraints on Days in Each Province

$n \geq 50$ $\quad$ $p \geq 60$

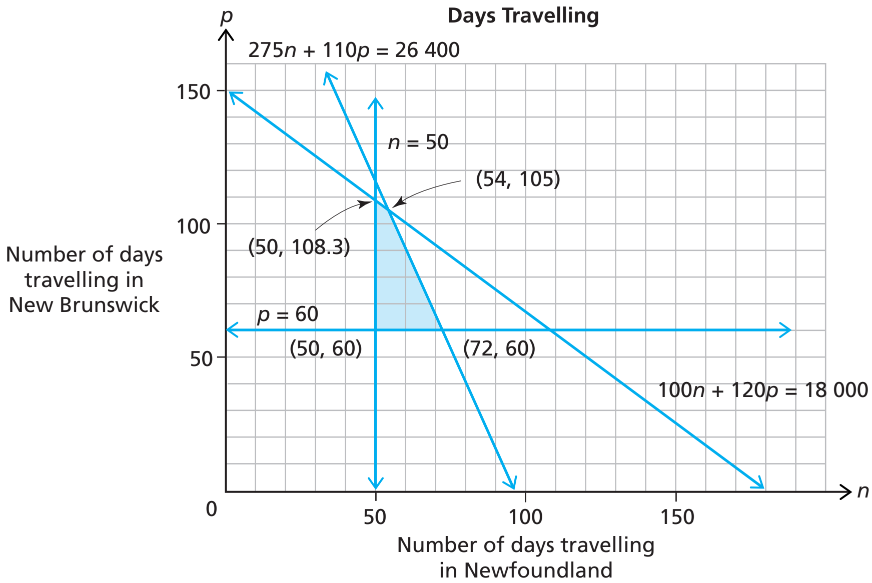
Constraints on Daily Travel Budget

$100n + 120p \leq 18\,000$

Constraints on Yearly Travel

$275h + 110p \leq 26\,400$

Constraints on Yearly Travel



(b) The constraints can be plotted on a graph and shading can be used to show the feasible region.

(c) Possible solutions can be estimated in the feasible region or on its borders. Some possible solutions are (60, 70), (72, 60), and (54, 105).

1.3 Connecting the Region and the Solution

The optimal solution to Vince's problem will occur at one of the four points that make up the vertices of the border on the feasible region. To find these vertices, there are four different systems of equations that need to be solved. As there are five possible methods that you have explored for solving systems of equations, four of them will be used here to solve Vince's problem.

– Note –
When you graph a feasible region, there are different lines that make up the border of the region. The vertices are the points where any two lines intersect. You can use your skills with solving systems of equations to find these vertices.

System 1 *Using the Graph*
n =50
$p = 60$
The solution to this system can be read accurately from the graph. The solution is $n = 50$ and $p = 60$.

System 2 *Algebraically*
$100n + 120p = 18\ 000$ ①
$n = 50$ ②

Rearrange equation ① to the form $y = mx + b$.

$$p = -\frac{5}{6}n + 150$$

Substitute $n = 50$.

The solution is $n = 50$ and $p = 108.333...$

System 3 *Table of Values*

$275n + 110p = 26\,400$ ①

$p = 60$ ②

Rearrange equation ① to the form $y = mx + b$.

$$p = -\frac{5}{2}n + 240$$

Create and enter a table of values for $p = -2.5n + 240$.
Look for a value of n that has a p value of 60.

n	p
40	140
50	115
60	90
70	65
80	40

Based on the table of values, the actual value of n will be between 70 and 80, since that is where the value of p changes from greater than 60 to less than 60.

The actual value of n that makes p equal to 60 is 72.

The solution is $n = 72$ and $p = 60$.

System 4 *Using Technology*

$100n + 120p = 18\,000$ ①

$275n + 110p = 26\,400$ ②

From ①

$100n = 18\,000 - 120p$

$p = -\frac{5}{6}n + 150$ ③

From ②

$p = -2.5n + 240$

Use the following steps to find the point of intersection on your graphing calculator.

- Press Y= to enter the equations.
- Enter the equations. For your calculator, x represents n and y represents p.
- Press 2nd: CALC:5 to show the graphs.
- Press ENTER when "First Curve?" appears on the screen.
- Press ENTER when "Second Curve?" appears on the screen.
- Press ENTER when "Guess?" appears on the screen. You will see a point of intersection of $p = 105$ and $n = 54$.

Historically, sales average \$3000 per day in Newfoundland and \$2800 per day in New Brunswick. The profit function, P, can be written as $P = 3000n + 2800p$.

Check the coordinates of the points found in the profit function to find the maximum profit. What is the maximum profit that Vince can make? How many days should he travel in each province?

Value of n	Value of p	$P = 3000n + 2800p$
50	60	\$318 000
54	105	\$456 000
50	108.333	\$453 333.33
72	60	\$384 000

Vince should travel 54 d in Newfoundland and 105 d in New Brunswick to maximize sales at \$456 000.

Example 3

Solve the system of equations.

$$x - 2y = 2 \quad ①$$
$$x + 2y = 4 \quad ②$$

Solution

From equation ①, $x = 2 + 2y$.
Substitute into equation ②.

$$\begin{aligned} 2 + 2y + 2y &= 4 \\ 2 + 4y &= 4 \\ 4y &= 2 \\ y &= \frac{1}{2} \end{aligned}$$

Substitute $\frac{1}{2}$ into equation ① to solve for x.

$$\begin{aligned} x - 2\left(\frac{1}{2}\right) &= 2 \\ x - 1 &= 2 \\ x &= 3 \end{aligned}$$

The solution to the linear system is $x = 3$ and $y = \frac{1}{2}$.

PRACTICE

1.1 Exploring an Optimization Problem

1. (a) Graph $y = 2x - 1$.

(b) Find three points on the graph that satisfy the inequality, $y \leq 2x - 1$.

(c) Graph $y \leq 2x - 1$.

2. Scott is a salesclerk at a local mall. Since he is the manager, his income is higher than that of the other staff. He is paid a base salary of $400 per week plus commission of 8% on his sales.

(a) Write an equation that represents Scott's income for one week.

(b) Write an inequality to represent the weekly incomes of the other employees.

(c) Write examples of possible weekly incomes for other employees.

(d) Construct a graph that represents all possible weekly incomes that are earned at the store. Are all of these incomes reasonable? Explain.

3. Graph each inequality. State at least one point that satisfies the inequality and one that does not.

(a) $y \geq 2x + 5$ (b) $y \leq x - 4$

(c) $y \geq -3x + 1$ (d) $y \leq -2x + 3$

(e) $y \geq \frac{1}{2}x - 4$ (f) $y \leq \frac{2}{3}x + 5$

4. Anna discovers that she is paying less for cable TV and Internet service than any of her friends. She pays $40 per month plus $2 per hour.

(a) Express Anna's costs as an equation.

(b) Express the amount that her friends pay as an inequality.

(c) Graph the inequality in part (b) and explain why you shaded as you did.

1.2 Exploring Possible Solutions

5. Graph each region.

(a) $y \geq 2x - 4$, $x \geq 0$, $y \geq 0$

(b) $y \geq \frac{1}{3}x - 2$, $y \leq 3x - 3$

(c) $2x - y - 3 \geq 0$, $x + y - 3 \leq 0$

(d) $y \leq -\frac{5}{2}x + 5$, $y \geq \frac{2}{5}x + 2$, $x \geq 0$, $y \geq 0$

(e) $3x + 2y + 12 \leq 0$, $3x + 5y - 50 \leq 0$

(f) $y \leq x + 4$, $y > -x + 1$, $x \geq 0$, $y \geq 0$

6. To manufacture cushions and pillows, a firm uses two machines, A and B. The time required on each machine is shown.

	Time on Machine A	Time on Machine B
Pillows	two minutes	four minutes
Cushions	nine minutes	seven minutes

- Machine A is available for one full shift of nine hours each day, and Machine B is available for parts of two shifts for a total of 10 h and 40 min each day.
- The profit on a cushion is $3.20.
- The profit on a pillow is $1.20.

(a) Write the constraints as inequalities.

(b) Choose three points and describe their meaning in relation to the constraints.

(c) Graph the constraints to form the feasible region.

(d) List and explain three feasible solutions to the problem.

7. Wendy makes stuffed toys to sell at the local market. Since this is her living, she has the following constraints.

- She can work a maximum of 40 h per week.
- It takes 2 h to make a small teddy bear and 4 h to make a large one.
- She can spend a maximum of $72 per week on materials.
- The materials for a small bear cost $3.
- The materials for a large bear cost $8.
- She makes a minimum of 4 small bears and 3 large bears per week.

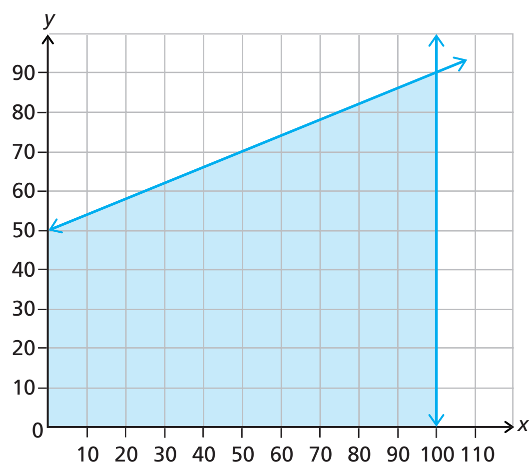
(a) Write the constraints as inequalities.

(b) Graph the constraints to form the feasible region.

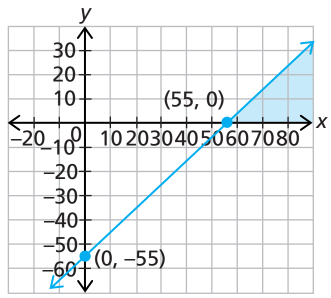
(c) List and explain three feasible solutions to the problem.

8. Find the inequalities that are represented on each graph.

(a)



(b)



1.3 Connecting the Region and the Solution

9. Casey Sports manufactures golf balls and tennis balls. The number of golf balls manufactured each day is represented by x in each pair of equations below. The number of tennis balls manufactured is represented by y. Find the point of intersection for each of the following. What does the point of intersection represent?

(a) $y = 3x - 3$
$y = -2x + 7$

(b) $y = -2x + 2$
$y = \frac{1}{2}x - 3$

(c) $y = 3x - 4$
$y = \frac{1}{2}x - 1$

(d) $y = -4x + 5$
$y = \frac{2}{3}x - \frac{13}{3}$

(e) $x - 5y = 10$
$3x + y = 16$

(f) $x = 6 - 3y$
$2x - y = 3$

10. A firm manufactures two-bulb bedroom lamps and four-bulb living room lamps.

- Each day, the manufacturer receives 480 bulbs.
- Each day, the manufacturer receives 180 lamp shades.
- Profit on a two-bulb lamp is $20 and profit on a four-bulb lamp is $35.

(a) Write each constraint as an inequality.

(b) Graph the constraints to form the feasible region.

(c) Find the vertices of the feasible region.

(d) Write the profit function and calculate the profit at each vertex.

(e) How many of each lamp should be made to yield a maximum profit?

11. Burlington Runners repairs sports shoes, particularly tennis shoes and jogging shoes. Two operators perform different functions to repair each shoe. The times required for each operation are given below.

- Tennis shoe: 16 min to strip and 12 min to re-sew
- Jogging shoe: 8 min to strip and 16 min to re-sew
- The two operators each work eight hours each day.
- The profits on a tennis-shoe repair are $5.00, and on a jogging-shoe repair are $3.00.

Ideally, how many pairs of each type of shoe should be repaired daily to maximize profits?

12. Jack and Carlene are planning to make punch to sell at a mall kiosk.

- They will use orange juice and fruit drink in their punch.
- Their recipe calls for at least twice as much orange juice as fruit drink.
- During the summer, they can sell between 12 L and 15 L of punch per day.
- They estimate that orange juice will cost $2.49 per litre, and fruit drink will cost $1.53 per litre.

How much of each juice should they buy to minimize the cost of the punch?

13. The Zimmer watch company manufactures two types of watches—a watch with hands (regular model) and a watch with a digital display.

- The digital model requires 1.5 h of machine time and 1 h of a jeweller's time.
- The regular model requires 30 min of machine time and 2 h of a jeweller's time.
- There are 3 h of machine time available each day.
- There are 7 h of jeweller time available per day.
- The profit on a digital model is $25 and on a regular model is $18.

How many of each type should be manufactured daily to achieve maximum profits?

14. A carpenter makes two types of bookcases, oak and maple. They are well made and she can sell as many as she can make.

- She can make a maximum of 20 bookcases per week.
- She can spend a maximum of $2400 per week on materials.
- The materials for an oak bookcase cost $100.
- The materials for a maple bookcase cost $150.
- An oak bookcase sells for $400 and a maple bookcase sells for $500.

How many of each kind should she make each week to maximize her income?