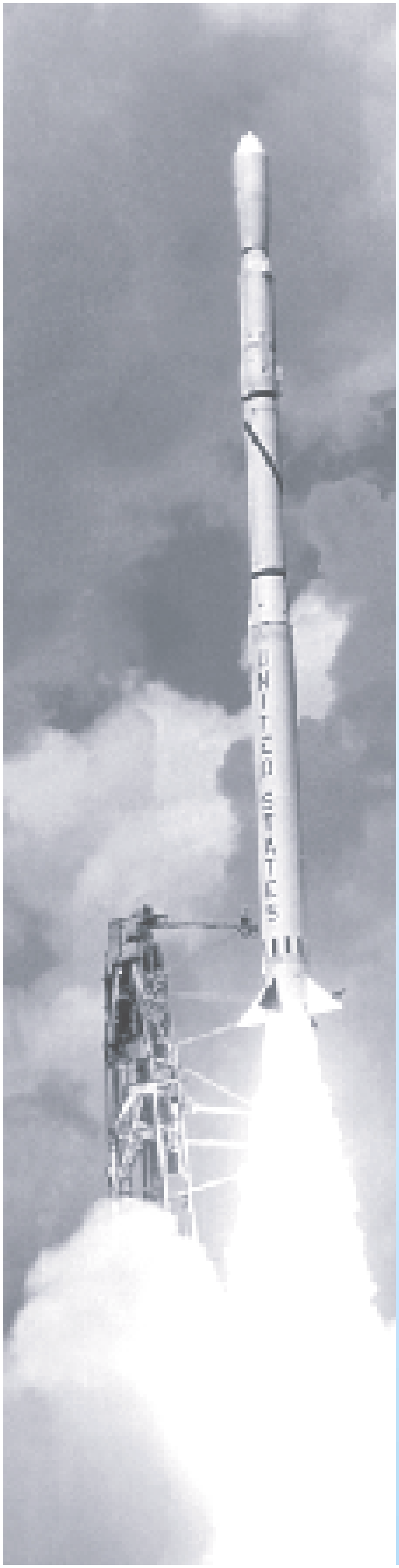


Chapter One

Quadratics

Many real-life situations involve non-linear relationships and functions. Businesses use quadratic relationships to maximize profits. Physicists use quadratic functions to describe the parabolic path of projectiles. In this chapter, you will look at ways to solve problems involving non-linear and, in particular, quadratic functions and relationships.

After successfully completing this chapter, you will be able to:

1. Distinguish between arithmetic and power sequences.
2. Model real-world phenomena using quadratic functions.
3. Describe general properties of quadratic and other non-linear relationships and visualize their graphs.
4. Represent quadratic functions in several ways and use these representations to describe the graphs of the functions.
5. Create scatter plots, determine the equations of the curves of best fit, and analyze the graphs using appropriate technology.
6. Use factoring and completing the square to determine the transformational form of a quadratic function.
7. Describe the properties of a parabola given the transformational form of a quadratic function.
8. Derive the transformational and standard forms of a quadratic function given the vertex and one other point of a parabola.
9. Derive, apply, and analyze the quadratic formula.
10. Use the roots of a quadratic equation to determine the number and existence of the *x*-intercepts of a parabola.
11. Use the discriminant to determine the nature of the roots of a quadratic equation.
12. Solve problems involving a quadratic equation.
13. Determine a quadratic function from a graph and from a table of finite differences.
14. Relate the sum and product of the roots of a quadratic equation to its coefficients.

1.1 Number Patterns

A contractor is building a chain-link fence containing 63 sections. If each section is a square made from four metal rods, and any two adjacent sections share one rod, how many rods are needed?

Investigation 1

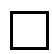
Number Patterns That Grow, Part 1

Purpose

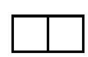
Explore some ways that number patterns can grow.

Procedure

A. Draw one square on grid paper to represent a fence formed with one metal square. Record the number of lines representing the number of metal rods as the first **term** of a **sequence**.



B. Add another square to represent a fence with two metal squares. Record the total number of metal rods as the second term of the sequence.



C. Continue this procedure until you have at least six terms in your sequence.

D. Examine the numbers in your sequence. What number patterns can you find?

E. Use a pattern to find the number of metal rods needed for the whole fence.

sequence—an ordered arrangement of numbers, symbols, or pictures in which each item or **term** follows another according to a rule

term—each item in a sequence. The symbol t_1 represents the first term in the sequence, t_2 represents the second term, and so on.

infinite sequence—a sequence that continues indefinitely and can be written as $\{t_1, t_2, t_3, \ldots\}$. An ellipsis (three dots) indicates the continuation of an infinite sequence.

finite sequence—a sequence that eventually terminates and can be written as $\{t_1, t_2, t_3, \ldots, t_n\}$ where the t_n represents the last or n^{th} term

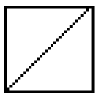
Investigation Questions

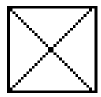
1. (a) Graph the value of the term, t_n, in your sequence versus the term number, n. Describe the shape of the graph.

(b) What are the domain and range of the graph?

(c) What is the slope of the graph?

2. Write a sequence to find the number of rods needed to build a 63-section fence using these shapes as sections placed side by side.

(a) 

(b) 

3. When rooms are built, drywall is put on the walls. Drywall can be cut into any shape. Design a room using a polygon shape other than a rectangle to represent each piece of drywall.

(a) Write a sequence with at least six terms to represent the number of studs needed to hold your drywall in place with from one to six sections.

(b) Have another student find the number of studs needed to hold your drywall in place with 100 sections.

4. Create your own sequence by selecting any number as the first term. Then add another number repeatedly to create a sequence of at least six terms. Exchange sequences with a classmate. Describe any patterns that you notice about this sequence.

5. (a) For each sequence in Questions 1 to 4, form a **sequence of differences** by subtracting the first term from the second term, the second term from the third term, and so on: $\{t_2 - t_1, t_3 - t_2, ..., t_n - t_{n-1}\}$.

(b) What do you notice about the terms in each sequence of differences in part (a)?

(c) Explain what you think a *common difference* means in sequences like those in Questions 1 to 4.

6. A bridge railing is to be formed by connecting sections of equilateral triangles. The contractor wants to find the number of rods represented by t_n, the n^{th} term of the sequence.
$\{3, 5, 7, ..., t_n\}$

Think about...

Questions 1 to 4

Why do you think each sequence is called an **arithmetic sequence**?

sequence of differences—a sequence created from another sequence by subtracting the value of each term in the original sequence from the next term in that sequence. For example, the sequence of differences for {1, 3, 5, 7, 9, …} is {3 – 1, 5 – 3, 7 – 5, …} or {2, 2, 2, …}.

(a) Explain why the terms t_1 to t_5 in the sequence can be expressed as follows:

$t_1 = 3 + 0 \times 2$
$t_2 = 3 + 1 \times 2$
$t_3 = 3 + 2 \times 2$
$t_4 = 3 + 3 \times 2$
$t_5 = 3 + 4 \times 2$

(b) Express the values of t_6 to t_{10} in a similar manner.

(c) Explain how to express t_n using a rule or function.

(d) How many metal rods are needed for a bridge railing constructed from 200 equilateral triangles?

(e) What are the domain and range of the function that you found in part (c)?

(f) Express t_n for any arithmetic sequence in general using a rule or a function.

CHALLENGE yourself

Prove that in any arithmetic sequence, the sequence of differences is a constant number.

Check Your Understanding

7. A decorative railing is to be constructed from sections in the shape of a regular hexagon.

(a) Create a sequence for t_1 to t_{10} representing the number of metal rods needed to build from one to ten sections.

(b) Create a rule or function to find the n^{th} term of the sequence representing the number of metal rods needed to construct a railing using n hexagonal sections.

(c) How many metal rods are needed to construct a railing using 200 regular hexagons?

8. Determine whether each sequence is arithmetic. If it is, use a rule or function to find the n^{th} term. If the sequence is not arithmetic, explain why.

(a) {–4, –8, –12, –16, ...}

(b) {3, 6, 9, 12, ...}

(c) {2, 7, 12, 17, ...}

(d) {1, 3, 9, 27, 81, 243, ...}

(e) {6.5, 8.5, 10.5, 12.5, 14.5, ...}

(f) {1, 4, 9, 16, 25, 36, ...}

(g) $\left\{\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, ...\right\}$

(h) {24, 19, 14, 9, ...}

CHALLENGE yourself

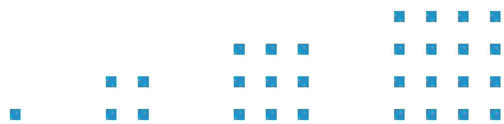
Prove that a linear function $t_n = an + b$ generates an arithmetic sequence for counting numbers n and real values of a and b.

9. Create an arithmetic sequence in which each term of the sequence of differences is

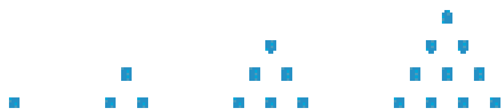
(a) –5 (b) –3.5 (c) $-\frac{3}{4}$ (d) –150

10. In each pattern below, four arrangements of dots are shown. For each pattern, write the number of dots in the arrangements as the first four terms in a sequence. If the sequence is arithmetic, explain why and write a rule or function to find the n^{th} term.

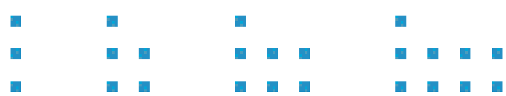
(a)



(b)



(c)



11. Use a rule or function to generate at least six terms of an arithmetic sequence. Exchange sequences with a classmate and write the rule or function used to generate the arithmetic sequence that you were given.

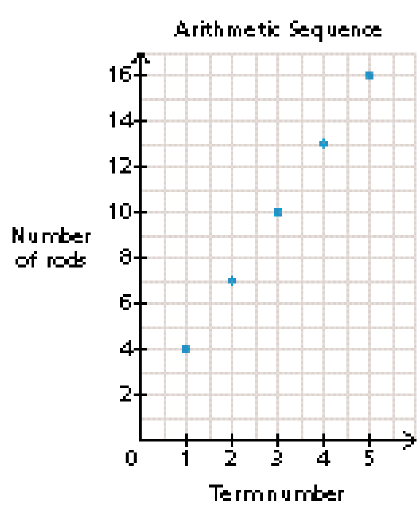
12. The function $t_n = 3n + 1$ in Investigation 1 generates the sequence {4, 7, 10, 13, 16, ...}. The values of the function can be represented in a scatter plot as shown.

(a) Explain how you know the graph represents an arithmetic sequence.

(b) If the points are joined, what is the slope of the graph?

(c) What is the difference between the successive terms in the sequence?

(d) How does this difference appear to be related to the slope of the graph of the function used to generate the sequence?



Did You Know?

Mathematicians in ancient Greece studied sequences formed by *figurate numbers*, or numbers that can be arranged in geometric figures or shapes. The representations in Question 10(a) and (b) are called *square numbers* and *triangular numbers* respectively.

13. Use each function below to generate the first six terms of a sequence.

$t_n = -3n + 2$ $t_n = -3n + 4.5$ $t_n = -3n - 0.5$

(a) How do you know is each sequence an arithmetic sequence?

(b) Graph each function. Why is each graph a straight line?

(c) Why is each line parallel to the other lines?

(d) How does the common difference appear to be related to the slope of the graph of each function used to generate an arithmetic sequence?

14. What is the common difference between the successive terms in the sequence generated by $t_n = -\frac{1}{3}n + \frac{1}{2}$?

15. Operating on the terms of an arithmetic sequence creates new sequences. Which sequences created by the following operations are still arithmetic sequences? How do you know?

(a) A constant number is added to each term.

(b) A constant number is subtracted from each term.

(c) Each term is multiplied by a constant number.

(d) Each term is divided by a constant number.

(e) Each term is squared.

16. Prove that an arithmetic sequence is created by

(a) adding or subtracting a constant number for each term of an arithmetic sequence

(b) multiplying or dividing each term of an arithmetic sequence by a constant number

CHALLENGE yourself

Create an arithmetic sequence for each given property. Write a linear function of the form $t_n = d(n - 1) + t_1$ to generate each term in the sequence.

(a) The fifth term is 12.

(b) The first term is 1 and the sixth term is 31.

(c) The first term is a negative integer and the fifth term is a positive integer.

(d) The common difference is a negative fraction.

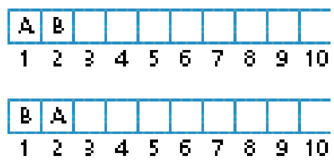
(e) The first term is your age and the eleventh term is 100.

Investigation 2

Number Patterns That Grow, Part 2

Each workday, Alice buys lunch from one of the 10 fast-food outlets in a mall, while her friend Beatrice buys lunch from a different fast-food outlet in the same mall. How many ways can they do this?

The diagrams show two different ways that Alice and Beatrice can buy lunch from Outlets 1 and 2.



Purpose

Explore other ways that number patterns can grow.

Procedure

A. Interpret the two diagrams above and summarize the results. Use similar diagrams to find the number of different ways Alice and Beatrice can buy lunch from Outlets 1 to 3. Use diagrams to find the number of ways that Alice and Beatrice can buy lunch from Outlets 1 to 4, and from Outlets 1 to 5.

B. Write each solution in Step A as a term in a sequence. Describe any patterns that you see.

C. Create a second sequence by finding the difference between successive terms in the sequence in Step B.

D. Create a third sequence by finding the difference between successive terms in the second sequence that you created in Step C. Describe what you notice about each term in this third sequence.

E. Use the results of Step C or D to solve the original problem. Explain what you did.

Think about...

Step C

Why do the results of Step C show that the sequence in Step B is different from the sequences in Investigation 1?

— Note —

The results of the subtraction of adjacent terms in a sequence $\{t_1, t_2, t_3, \ldots\}$ are referred to as the sequence of **first-level differences**. These differences form a new sequence, $\{t_2 - t_1, t_3 - t_2, \ldots\}$, represented by D_1. In a similar way, the results of the subtraction of adjacent terms in the sequence D_1 are referred to as the sequence of **second-level differences**, represented by D_2. The results of further subtractions of adjacent terms are called sequences of **third-level differences** (D_3), sequences of **fourth-level differences** (D_4), and so on.

Investigation Questions

17. Alice used the following reasoning to determine the number of different ways that she and Beatrice can buy different types of lunches from 10 food outlets.

Alice reasoned that she has 10 choices to buy lunch from any one of the 10 outlets.

A	A	A	A	A	A	A	A	A	A
1	2	3	4	5	6	7	8	9	10

Once Alice chooses a food outlet, Beatrice has only nine choices if she wants to buy lunch from a different outlet. For example, if Alice chooses Outlet 1, Beatrice must choose from the remaining nine outlets.

A	B	B	B	B	B	B	B	B	B
1	2	3	4	5	6	7	8	9	10

There are $10 \times 9 = 90$ possible ways for Alice and Beatrice to purchase different types of lunches from 10 food outlets.

(a) Write a function to show the number of ways for two people to purchase different types of lunches from n food outlets.

(b) Use the function to find the number of ways for two people to purchase different types of lunches from 50 food outlets.

18. In a round-robin tournament, each team plays each other team only once.

(a) Starting with two teams, create a sequence with five terms to show the relationship between the number of teams in a round-robin tournament and the number of games that need to be scheduled.

(b) Compare the sequence in part (a) to the sequence in Step B of Investigation 2. Tell what you notice.

(c) Use the results of part (b) to determine the number of games if there are 10 teams in the round-robin tournament.

(d) Write a function to determine the number of games if there are n teams.

19. Explain how the function in Question 18(d) would be modified if the tournament scheduled three games between each team and every other team.

20. (a) Graph each function from Questions 17, 18, and 19. Describe each shape.
(b) Explain how the shape of the graph shows you that each function does not generate an arithmetic sequence.
(c) Reconstruct the graphs by extending the domain of each graph in part (a) to include real numbers instead of only positive integers.
(d) What is similar about the graphs in parts (a) and (c)? What is different?

21. (a) Explain why each function from Questions 17, 18, and 19 is a **quadratic function**.
(b) What numerical value is the **coefficient** a of n^2 in each quadratic function?
(c) How does this coefficient compare with the common differences in each sequence generated by each function?

Check Your Understanding

22. How do arithmetic and quadratic sequences differ?

23. (a) Use each quadratic function below to generate a **quadratic sequence** with ten terms.

$t_n = n^2 + 2n - 2$ $\quad t_n = 2n^2 + 2n - 1$ $\quad t_n = \frac{1}{3}n^2 + 3n$

$t_n = -4n^2 + 2n$ $\quad t_n = -0.2n^2 - 1$

(b) Find the sequence of first-level differences, D_1, and the sequence of second-level differences, D_2, for each sequence in part (a).
(c) Compare each term in the sequence D_2 with the coefficient a in each function. Describe what you notice.
(d) Use the results from part (c) to predict the sequence of second-level differences, D_2, in the sequence generated by the quadratic function $t_n = -\frac{5}{3}n^2 + 2n - 1$.

24. Use a quadratic function to generate a sequence with at least ten terms. Exchange sequences with another student. Use the values in the sequence of second-level differences, D_2, to predict the coefficient a in the quadratic function used by the other student.

25. The quadratic function $t_n = an^2 + bn + c$ is used to generate a quadratic sequence. Prove that each term in the sequence of second-level differences, D_2, equals $2a$, where a is the coefficient of n^2 in the quadratic function.

quadratic function—a function that can be represented by $y = ax^2 + bx + c$, where a and b are **coefficients**, $a \neq 0$, and c is a constant number. Because the greatest exponent in the function is 2, the function is said to have a **degree** of 2.

coefficient—the constant part of a term in an expression; for example, in the expression $2x^2 + 3x - 9$, 2 is the coefficient in the term $2x^2$

quadratic sequence—a sequence whose terms are generated by a quadratic function

Think about...

Question 26(a)

Show that the cubic function in part (a) is equivalent to the cubic function $t_n = n^3 - 3n^2 + 2n$. What is the degree of a cubic function?

26. In Question 17, a quadratic function representing the number of ways for two people to buy different types of lunches at a mall with n food outlets was created.

(a) Use similar reasoning to create a function to find the number of ways for three people to buy different types of lunches at a mall with n food outlets.

(b) Why do you think the function in part (a) is called a *cubic* function?

(c) Use the function in part (a) to find the number of ways for three people to buy different types of lunches at a mall with 10 food outlets.

(d) Graph the cubic function in part (a) for a domain of all real values. How does the graph of a cubic function compare to the graph of a quadratic function?

(e) Use the cubic function to generate ten terms for n from 3 to 12.

(f) Find the sequences of differences, D_1, D_2, and D_3.

(g) What do you notice about D_3?

27. (a) Generate ten terms for each sequence by substituting values for n from 1 to 10 in each cubic function.

$t_n = 2n^3 + n + 1$ $\qquad$ $t_n = 3n^3 + n$

$t_n = -4n^3 + n + 1$ $\qquad$ $t_n = \frac{1}{6}n^3 + n + 1$

(b) Find the sequences of differences, D_1, D_2, and D_3, for each sequence in part (a).

(c) Describe what you notice about D_3.

(d) Compare each term in D_3 with the coefficients in each cubic function. Describe what you notice.

28. A sequence is generated by the cubic function $t_n = an^3 + bn^2 + cn + d$ for the positive integers. Demonstrate that each term in the sequence of third-level differences, D_3, is the constant $6a$.

29. (a) Explain why you think a sequence generated by the function $t_n = 2n^4 - 2$ is called a *quartic* sequence.

(b) Use the results of Question 27(c) to make a prediction about the sequence of differences D_4 for a quartic sequence. Verify your prediction by using the quartic function to generate at least ten terms.

(c) Suppose the function in part (a) were changed to $t_n = 2n^5 + 4n^3 - 5n - 3$. For which sequence of differences would the terms likely have a constant value?

CHALLENGE yourself

Show how the problem of finding the number of ways to seat four people in n seats can be developed into a quartic function.

30. (a) Each sequence in Investigation 2 is called a *power sequence*. Write your own definition of a power sequence.

(b) What is the relationship between the sequence of differences, D_n, and the degree of the function generating a power sequence?

31. Generate any type of power sequence containing at least ten terms. Exchange sequences with another student. Use sequences of differences to find the type of power sequence used by the other student.

32. Form a new sequence of ten terms by multiplying the corresponding terms of two arithmetic sequences.

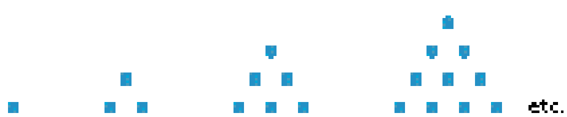
(a) What type of sequence is obtained?

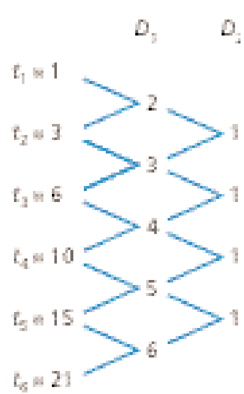
(b) Explain why this type of sequence is obtained.

33. Prove that multiplying the corresponding terms of two arithmetic sequences results in a quadratic sequence.

FOCUS A Creating Quadratic Functions

The number of dots in successive triangular numbers forms sequences of differences D_1 and D_2 as shown below.





The common difference, which is each term in the sequence of second-level differences, D_2, and equation solving can be used to find the function t_n used to generate the quadratic sequence.

CHALLENGE yourself

Create a sequence with each given property:

(a) A quadratic where the first term is 3

(b) A cubic where each term in D_3 is negative

(c) A quartic where the first term is a negative integer

(d) A quadratic where the second term is $\frac{1}{4}$

(e) A quadratic where every term is a negative integer

Think about...

The Sequences of Differences

Explain how you know that

(a) the sequence is not an arithmetic sequence

(b) the sequence is quadratic

(c) the coefficient *a* in the quadratic function equals $\frac{1}{2}$

In general, a quadratic sequence is generated by the function $t_n = an^2 + bn + c$. You discovered in Investigation 2 that each term in D_2 is equal to $2a$. Therefore, $a = \frac{1}{2} \times \left(\text{each term in } D_2.\right)$

For the triangular-number sequence above, each term in D_2 is 1. Therefore, $a = \frac{1}{2}$. Substituting $a = \frac{1}{2}$ in $t_n = an^2 + bn + c$, and using the values of 1 for t_1 and 3 for t_2, produces the following equations:

$$1 = t_1 = \frac{1}{2}(1)^2 + b(1) + c \qquad \text{Equation ①}$$

$$3 = t_2 = \frac{1}{2}(2)^2 + b(2) + c \qquad \text{Equation ②}$$

Focus Questions

34. (a) Show how to simplify Equations ① and ② to produce these two linear equations:

$$\frac{1}{2} = b + c$$

$$1 = 2b + c$$

(b) Solve each equation for b and c.

35. Use your solutions in Question 34 to create a quadratic function that will produce the sequence of triangular numbers for positive integers n.

36. Explain why the quadratic function in Question 35 is equivalent to

$$t_n = \frac{n(n+1)}{2}.$$

37. Use the quadratic function to find the 20th triangular number.

Think about...

Choosing Terms t_1 and t_2

The values of b and c were found by substituting the values of t_1 and t_2. Substitute the values of two other terms instead of t_1 and t_2. Do you still obtain the same quadratic function?

CHALLENGE yourself

Each term in a sequence of second-level differences, D_2, is –3. Use this information to create a quadratic function in which

(a) $b = 0$ and $c = 0$

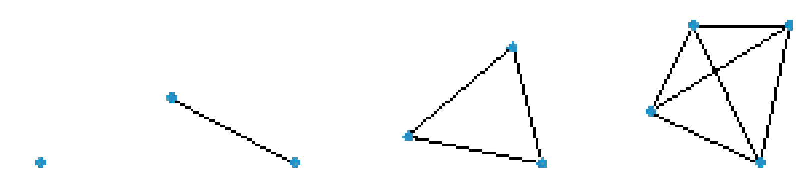
(b) $b \neq 0$ and $c = 0$

(c) $b = 0$ and $c \neq 0$

(d) $b \neq 0$ and $c \neq 0$

Check Your Understanding

38. Each diagram shows the number of line segments needed to connect a set of n points, no three of which lie in a straight line.



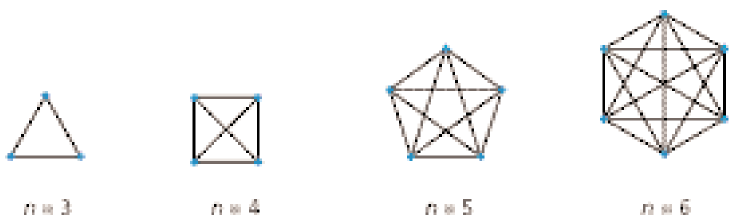
n = 1 n = 2 n = 3 n = 4

(a) Create a sequence with at least six terms to show the relationship between the number of points and the number of line segments needed to connect every point to every other point.

(b) Create a quadratic function $t_n = an^2 + bn + c$ to generate the sequence in part (a).

(c) Use the function in part (b) to determine the number of line segments needed to interconnect 25 points, no three of which lie in a straight line.

39. Diagonals are formed on regular polygons, starting with a three-sided regular polygon or equilateral triangle. The variable n represents the number of sides in the polygon.



Determine the number of diagonals in a regular polygon with 20 sides. Then determine the equation for the number of diagonals in a regular polygon of n sides.

40. The following sequence shows the height in metres above the ground of a falling rock after 0, 1, 2, 3, 4, 5, and 6 s:
{1000, 995.1, 980.4, 955.9, 921.6, 877.5, 823.6}.

(a) Explain why the sequence is a quadratic sequence.

(b) Create a quadratic function $t_n = an^2 + bn + c$, where t_n represents the height in metres of the falling rock after n seconds.

(c) Use the function in part (b) to determine when the rock hits the ground.

41. A Frisbee is thrown straight up in the air from a height of 2 m. The height of the Frisbee in metres after each of 0, 0.2, 0.4, 0.6, 0.8, and 1 s is shown.
{2.00, 3.12, 4.08, 4.88, 5.52, 6.00}

(a) Explain why the sequence is a quadratic sequence.

(b) Create a quadratic function $t_n = an^2 + bn + c$, where t_n represents the height in metres of the Frisbee after n seconds.

(c) Use the function in part (b) to determine when the Frisbee hits the ground.

CHALLENGE yourself

Five pennies and five dimes are arranged on square paper.

A coin can move forward one space to an empty square or jump over another coin to an empty square. Only one coin can occupy a square. Switch the pennies with the dimes in as few moves as possible.

(a) Create a sequence of the minimum number of moves needed for one, two, three, four, and five pairs of coins. Identify any patterns.

(b) Use a sequence of second-level differences to identify the type of sequence and to determine a function for the minimum number of moves for n pairs of coins.

(c) Use your function in part (b) to determine the minimum number of moves for ten pairs of coins.

42. At an airshow, a plane is in a power dive. The height of the plane in metres after 1, 2, 3, 4, 5, and 6 seconds is shown.
{71, 64, 59, 56, 55, 56}

(a) Explain why the sequence is a quadratic sequence.

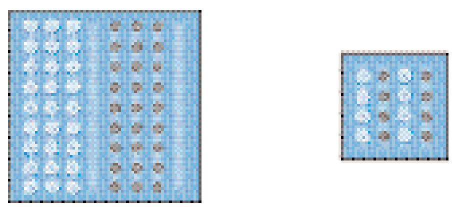
(b) Create a quadratic function $t_n = an^2 + bn + c$, where t_n represents the height in metres of the plane after n seconds.

(c) Use the function in part (b) to determine the minimum height of the plane.

Chapter Project

Fencing Gardens

A gardener has 100 m of fencing with which to enclose two square garden plots. The fences must not have a common side. To form each square fence, she cuts the fencing into two lengths.



(a) Create a table showing the relationship between the perimeter of one plot and the total area of both square plots.

(b) Use information about sequences of differences to determine the relationship between the perimeter of one plot and the total area of both square plots.

(c) Determine the relationship between the lengths used to form each fence and the total area of both garden plots if each is formed into a circle instead of a square.

1.2 Non-linear Relationships and Functions

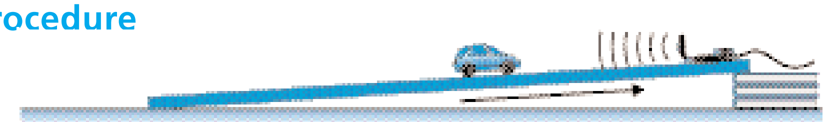
Investigation 3

Analyzing Non-linear Data

Purpose

In this Investigation, you will collect and analyze non-linear data.

Procedure



A. Form an angled ramp by placing two or three books under one end of a 1.2- to 3-m board or plastic eavestrough. The angle between the ramp and the floor should be only a few degrees, so that when a toy car or ball is pushed up the ramp, it slows, stops, then rolls back down the ramp slowly. A measuring device should be placed at the top of the ramp and connected to a graphing calculator. The device will measure the distance between itself and the car or ball (travelling unassisted) at very short time intervals.

B. Have someone activate the calculator while another person pushes the car or ball so that it travels up, stops, and travels back down the ramp. After the measuring device has finished collecting the data, the calculator will display a graph of time versus distance.

Investigation Questions

1. Compare the graph of the data to your graph from Think about ... Step B, part (a).

2. (a) Use the shape of the graph and/or the common differences to decide whether linear, quadratic, or cubic regression should be used to find the equation of the curve of best fit.

(b) Use TRACE or LIST to examine time data and distance data. Record the data in a table.

(c) Use the data in part (b) to find the equation of the curve of best fit.

Think about...

Step B

When the car or ball rolls up, stops, and rolls back down the ramp, it travels a distance over a period of time. Sketch two graphs of time (x-variable) versus distance (y-variable) that you think represent this relationship between time and distance.

(a) For the first graph, use the distance y to represent the distance from the measuring device.

(b) For the second graph, use the distance y to represent the distance from the starting position of the ball or toy car.

(c) Describe the shape of each graph. Explain your reasoning.

3. Suppose the experiment were repeated using the same force but on a much steeper ramp than in Investigation 3. Predict the shape of the graph of time versus distance travelled by a ball or car as it rolls up and down a steeper ramp.

4. (a) Use the data in the table from Question 2(b) to create a table of time and distance from the starting point.

(b) Use the data in the table to construct a graph of the distance from the starting point over time.

(c) Compare your graph to the graph that you sketched in Think about ... Step B, part (b).

(d) Find the equation of the curve of best fit.

Check Your Understanding

The Giant Drop ride is located at Dreamworld in Australia's Gold Coast. Riders enter a gondola that is raised to 119 m (or 39 storeys). When the gondola is dropped, it accelerates to a velocity of 135 km/h before a magnetic braking system stops it, metres from the ground. The table shows the time in seconds and approximate drop distance in metres.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Drop (m)	0.00		4.90		19.60		44.10		78.40		

Think about...

The Giant Drop Ride Data

Explain how the data show that the object is not falling at a uniform or constant velocity.

Explain how calculating sequences of differences D_1 and D_2 can help you find the equation of the curve of best fit.

5. (a) Construct a scatter plot of time versus drop distance.

(b) Use the shape of your graph and/or sequences of differences to determine whether linear, quadratic, or cubic regression should be used to find the equation of the curve of best fit.

(c) Find the equation of the curve of best fit.

6. Use the equation of the curve of best fit from Question 5.

(a) Complete the table.

(b) Determine whether the gondola falls twice the vertical distance in twice the time. Explain your reasoning.

7. (a) Use the data in the table from Question 6 to calculate the average velocity of the gondola in metres per second by dividing each drop distance by the time it takes to reach that distance.

(b) Create a scatter plot of average velocity in metres per second and time in seconds, where the average velocity is the dependent variable y and time is the independent variable x. Describe the shape of your graph.

(c) Use the shape of your graph in part (b) to determine whether linear, quadratic, or cubic regression should be used to find the equation of the curve of best fit.

(d) Find the equation of the curve of best fit.

8. The rides Superman the Escape and Tower of Terror accelerate riders to 160 km/h in 7 s, then raise them to a vertical height of 40 storeys. On the reverse route, riders travel backward up to 160 km/h again before magnetic brakes bring them to a complete stop.

The table shows the horizontal distance travelled in each second of acceleration.

Time (s)	1	2	3	4	5	6	7
Distance (m)	3.17	12.70	28.57	50.79	79.36	114.27	155.54

(a) Find the equation of the curve of best fit.

(b) Find the equation of the curve of best fit using a different method than in part (a).

9. The table shows the height of a football (in metres) after various lengths of time measured in seconds.

Time (s)	1	2	3	4	5
Height (m)	25.1	38.4	41.9	35.6	19.5

(a) Find the equation of the curve of best fit.

(b) At what height was the ball initially thrown? How do you know?

(c) How long does it take the ball to hit the ground?

FOCUS B Comparing Linear and Non-linear Relationships

A driver of an automobile is travelling at 110 km/h on an asphalt road in good weather conditions. He spots a family of deer standing on the road at a distance of 100 m (or about the length of a football field). If he applies the brakes immediately, will he be able to stop in time to avoid striking the deer?

The estimated braking distances and perception–reaction distances (distances travelled in the time between when a driver spots a hazard and when the driver reacts) at various speeds on various surfaces are shown on the next page.

Did You Know?

Which form of North American wildlife is the most dangerous? bears? cougars?

It might surprise you to know that deer are considered the most dangerous wildlife in North America because of collisions with motor vehicles. Accidents in which vehicles hit wildlife are no small matter in British Columbia. In 1999, more than $11 million in BC insurance claims was paid out for the 6411 incidents of this sort.

Speed (km/h)	Braking distance (m)				Perception–reaction distance (m)
	Snow/ ice	Light frost	Gravel	Asphalt	
0	0.00	0.00	0.00	0.00	0.00
10	2.61	1.06	0.78	0.57	4.17
20	10.49	5.00	3.14	2.25	8.33
30	23.61	11.56	7.08	5.07	12.50
40	41.99	20.75	12.59	9.00	16.67
50	65.62	32.56	19.68	14.06	20.83
60					
70					
80					
90					
100					

As you discovered in Section 1.1, linear relationships can be identified by a constant growth rate. An equal increase or decrease in the independent variable in a linear relationship results in an equal increase or decrease in the dependent variable. For example, the speed and the perception–reaction distance shown in the table form a linear relationship, because each time the speed increases or decreases by 10 km/h, the perception–reaction distance increases or decreases by about 4.17 m.

Speed (km/h)	Distance (m)	Increase in distance (m)
10	4.17	
20	8.33	4.16
30	12.5	4.17
40	16.67	4.17

Non-linear relationships such as quadratic ones do not have a constant or uniform growth rate.

Think about...

The Constant Increase

Explain how the constant increase of 4.17 can be used to find the slope of the line for a graph of speed versus perception–reaction distance.

Explain how the perception–reaction distance data can be used to estimate the average time it takes a driver to be aware of a traffic obstruction and to begin to apply the brakes.

Focus Questions

10. Use any techniques that you have learned to identify the relationship between the speed and the braking distance on any road surface and to find each equation of the curve of best fit.

11. Use each equation of the curve of best fit from Question 10 to complete braking distances in the table.

12. Use any techniques to complete the perception–reaction distances in the table.

13. Use the equations of the curves of best fit found in Question 10 to estimate the braking distance for a car travelling at a legal highway speed of 110 km/h and an illegal highway speed of 200 km/h on each type of surface.

14. Solve the problem at the beginning of Focus B about stopping before the family of deer.

15. What is the relationship between doubling the speed of a car and

(a) the braking distance on each surface?

(b) the perception–reaction distance?

16. Suppose the perception–reaction distance is added to the braking distance of each surface. Describe the shape of the graph of speed in kilometres per hour and total stopping distance in metres.

Check Your Understanding

17. The severity of an automobile crash increases significantly as the speed increases. The table shows that the severity of a crash at 20 km/h is four times as great as that of a crash at 10 km/h.

Speed (km/h)	10	20	30	40	50	60	70	80	90	100
Crash severity index	1	4	9	16	25	36	49	64	81	100

(a) Is the relationship between crash severity and speed quadratic or linear? Explain.

(b) How would the crash severity increase for a head-on collision with a vehicle of the same mass travelling at the same speed?

(c) What would the crash severity index be for a vehicle travelling at 200 km/h?

CHALLENGE yourself

Various Web sites relate braking distance and speed, but many sites use Imperial instead of metric measurement. For example, the US Department of Transportation site (**www.fhwa.dot.gov/tndiv/saftrain.htm**) presents this table on braking distances:

	30 mph	50 mph	60 mph
Automobile	76 ft	174 ft	238 ft
Large truck	98 ft	223 ft	303 ft
Freight train	3150 ft	7000 ft	8500 ft

(a) Construct a table of speeds in kilometres per hour and braking distances in metres.

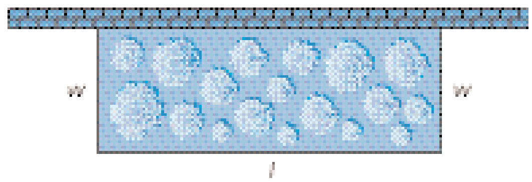
(b) Use the table to find the curves of best fit for each type of vehicle.

(c) A car travelling at 70 km/h is 75 m from a railway crossing. A freight train travelling at 50 km/h is 300 m from the crossing. If the driver and engineer notice each other at the same time and both take 1.5 s to start braking, will they collide?

Investigation 4

Finding Maximum and Minimum Values

A gardener uses 50 m of bordering to form a rectangular garden. One side of the garden touches a wall and doesn't need any bordering. What length and width of the garden result in it having the greatest area?



Purpose

Use modeling and graphing to determine a maximum area.

Procedure

A. Calculate the length and area of the garden for each width shown in the table.

Width (m)	5	6	7	8	9	10	11	12	13	14	15	16
Length (m)												
Area (m^2)												

B. Describe any patterns that you notice in the table.

C. Use any techniques that you have learned to identify the relationship between the width and area of the garden and to find the equation of the curve of best fit.

Investigation Questions

18. Graph the equation of the curve of best fit from Step C for widths from zero to 25 m. Explain how the graph can be used to estimate the maximum area of the rectangular garden.

19. Examine the diagram of the rectangular garden.

(a) Express the length, l, as a function of the width, w, and the total length of the bordering.

(b) Use the expression $A = lw$ and the expression in part (a) to create a function in which A is the dependent variable and w is the independent variable.

(c) Identify the type of function you created in part (b).

Think about...

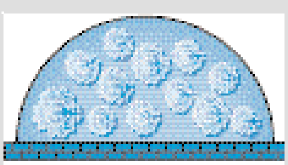
Steps A and B

Describe the shape of the garden as the area approaches zero.

Explain how you know that the relationship between width and length is linear but the relationship between width and area is non-linear.

CHALLENGE yourself

A 50-m length of bordering is formed into a semicircle with a wall forming the diameter.



Would a semicircular garden have more or less area than the maximum area of the rectangular garden in Question 18? Explain.

(d) Show that the function in part (b) is equivalent to the equation of the curve of best fit that you found in Step C in Investigation 4.

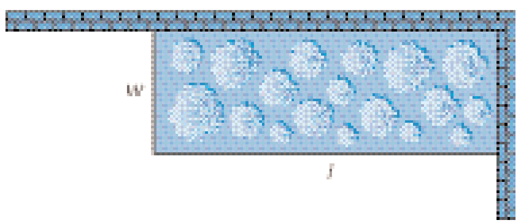
Check Your Understanding

20. Suppose the gardener formed the garden using 100 m of bordering instead of 50 m.

(a) Express the area of the garden as a function of its width.

(b) Use the graph of the function in part (a) to estimate the length and width that maximize the area of the garden.

21. Suppose the gardener used two adjacent sides of the wall and 50 m of bordering to form the garden.

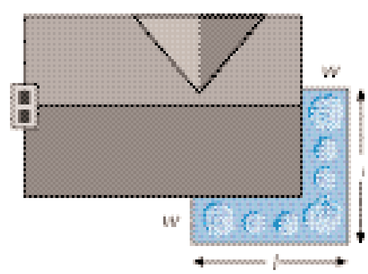


(a) Express the area of the garden as a function of its width.

(b) Use the graph of the function in part (a) to estimate the length and width that maximize the area of the garden.

(c) What are some advantages and disadvantages of this gardening arrangement?

22. Suppose the 50 m of bordering was placed at the corner of a building to form an L-shaped garden.



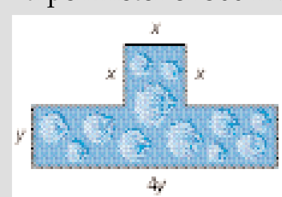
(a) Express the area of the garden as a function of its width.

(b) Use the graph of the function in part (a) to estimate the length and width that maximize the area of the garden.

(c) What are some advantages and disadvantages of this arrangement?

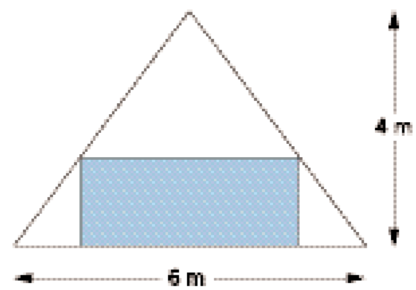
CHALLENGE yourself

The garden below has a perimeter of 300 m.



Find a value for x and a value for y that produce a garden with minimum area.

23. An isosceles triangle has a base of 6 m and a height of 4 m.



(a) Find the maximum area and dimensions of a rectangle within the triangle as shown in the diagram.

(b) Repeat part (a) for a rectangle inside an isosceles triangle with a base of 4 m and a height of 12 m.

(c) Compare the dimensions of the base and height of each isosceles triangle with the dimensions of each rectangle found in parts (a) and (b). What do you notice?

24. Find two numbers whose sum is 100 and whose sum of squares is a minimum. How do you know it is a minimum?

25. In a newspaper contest, a problem is posed:
The last two numbers for a combination to open a safe add up to 44. The product of the two numbers is a maximum. What are the last two numbers?

26. A piece of gold braid, 60 cm in length, is to be cut into two pieces to form two squares. If the area of each square is to be a minimum, what are the dimensions of the squares?

27. The coordinates of points P and Q are (1, 2) and (2, –3), respectively, and R is a point on the line $x = -1$. Find the coordinates of R so that $PR + RQ$ is a minimum.

Chapter Project

Fencing Gardens

In Section 1.1, you investigated and identified the type of relationship between the variable x, the first length of fencing, and the total area A of two *square* garden plots formed from 100 m of fencing.

(d) Use the data collected in Section 1.1 to determine the equation of the curve of best fit, and use the equation to determine the value of x that minimizes the area of each type of garden.

(e) Students in the enhanced course also investigated the relationship between the lengths used to form each fence and the total area of both garden plots if each is formed into a *circle* instead of a square. Use another method to create each equation.

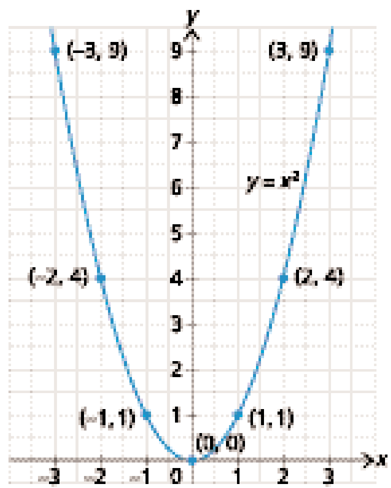
1.3 Properties of Graphs of Quadratic Functions

Focus C: Forms of Quadratic Functions

parabola—the curved graph of a quadratic function

transformational form of a quadratic function—a quadratic function expressed as $\frac{1}{a}(y - k) = (x - h)^2$, for which all variables are real numbers and $a \neq 0$

The graph of any quadratic function is a curve called a **parabola**. In Sections 1.1 and 1.2, you constructed several parabolas. In this focus, you will be reminded that any quadratic function is simply a transformation of the basic quadratic function $y = x^2$. You have seen functions written in **transformational form** in *Mathematical Modeling, Book 1*. You will also discover that different forms of the equation of a quadratic function provide different information about the parabola.



vertex of a parabola—the point on a parabola where a minimum or a maximum *y*-value occurs

axis of symmetry—a line in which a parabola or other graph is reflected onto itself

This parabola is the graph of the basic quadratic function $y = x^2$. The function has a minimum *y*-value of 0 at the origin, (0, 0). The point on a parabola where a minimum or a maximum *y*-value occurs is called the **vertex of a parabola**.

As a point on a parabola moves away from its vertex, a pattern is formed. The following pattern can help you sketch the graph of $y = x^2$.

$y = x^2$ over 1 up 1, over 2 up 4, over 3 up 9, over 4 up 16

The parabola $y = x^2$ has reflection or line symmetry because a mirror placed on the line or **axis of symmetry,** whose equation in this case is $x = 0$ (the *y*-axis), will reflect each point on the parabola onto an opposite point on the parabola. Every parabola has an axis of symmetry.

Because the parabola has a minimum *y*-value, it *opens upward*.

Think about…

A Parabola

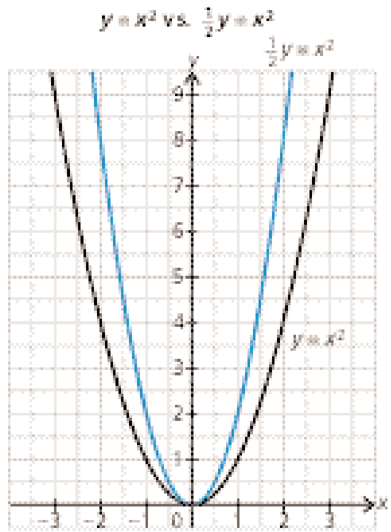
Describe the properties of a parabola that *opens downward*.

Example 1

Compare the graphs of $y = x^2$ and $\frac{1}{2}y = x^2$. How are they alike? How are they different? Complete each example before answering these two questions.

Solution

A table of values and the graph of the functions can help you compare them. The table of values is shown in the margin.



Each y-value of the parabola of $\frac{1}{2}y = x^2$ is twice the y-value of the parabola of $y = x^2$. The function $\frac{1}{2}y = x^2$ is said to have a **vertical stretch** of 2 of the function $y = x^2$. As a point on each parabola moves away from its vertex, a pattern is formed.

- $y = x^2$ — over 1 up 1, over 2 up 4, over 3 up 9, over 4 up 16
- $\frac{1}{2}y = x^2$ — over 1 up 2, over 2 up 8, over 3 up 18, over 4 up 32

When you compare the patterns for $y = x^2$ and $\frac{1}{2}y = x^2$, you can see that the vertical stretch factor is always 2. As a mapping rule, this can be represented as $(x, y) \rightarrow (x, 2y)$.

– Note –

The table of values for the parabolas in Example 1 is shown below. The values in the second and third column show the y-values of each function for the given value of x.

x	$y = x^2$	$\frac{1}{2}y = x^2$
–3	9	18
–2	4	8
–1	1	2
0	0	0
1	1	2
2	4	8
3	9	18

vertical stretch—a ratio that compares the change in y-values of a parabola of a quadratic function with the corresponding y-values of the parabola of $y = x^2$

Sometimes, quadratic functions are written in other forms. Two of these are **standard form** and **general form**. In the case of the function $\frac{1}{2}y = x^2$, the standard form and the general form can both be written as $y = 2x^2$. Before looking at the next example, answer the following questions. Keep them with you as you complete the Focus.

- In general form, what is the value of a in $y = 2x^2$?
- In general form, what is the value of b in $y = 2x^2$?
- In general form, what is the value of c in $y = 2x^2$?
- What does the value of a in the general form of a quadratic function tell you? What does the value of c tell you?

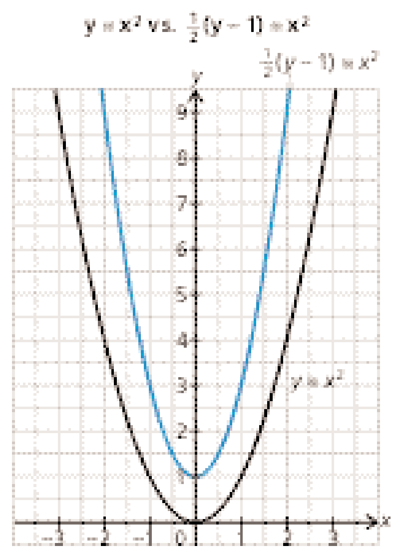
Example 2

Compare the graphs of $y = x^2$ and $\frac{1}{2}(y - 1) = x^2$. How are they alike? How are they different? As in Example 1, answer these questions on your own when you have completed the Focus.

Solution

As in Example 1, a table of values and the graph of the two functions can help you compare them.

x	–3	–2	–1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$\frac{1}{2}(y - 1) = x^2$	19	9	3	1	3	9	19



standard form—a quadratic function written in the form $y = a(x - h)^2 + k$ where $a \neq 0$

general form—a quadratic function written in the form $y = ax^2 + bx + c$, where $a \neq 0$

Think about...

The Form of the Function

Explain how to change the function $\frac{1}{2}(y - 1) = x^2$ so it can be graphed using a graphing calculator.

Think about...

Measuring Vertical Stretch

Explain why each vertical distance for $\frac{1}{2}(y - 1) = x^2$ is measured from the line $y = 1$, not the line $y = 0$ (the x-axis).

– Note –

Remember how to read the table of values from Example 1. The second and third rows give the y-value of each function for the given value of x.

The vertex of the parabola for $\frac{1}{2}(y - 1) = x^2$ is located at (0, 1), while the vertex of the parabola for $y = x^2$ is located at (0, 0). This shows that the function $\frac{1}{2}(y - 1) = x^2$ has a *vertical translation* of 1 when compared to the function of $y = x^2$. As a point on each parabola moves away from its vertex, a pattern is formed.

- $y = x^2$ over 1 up 1, over 2 up 4, over 3 up 9, over 4 up 16
- $\frac{1}{2}(y - 1) = x^2$ over 1 up 2, over 2 up 8, over 3 up 18, over 4 up 32

The patterns show that, for each unit change in x from zero, the change in the y-value of $\frac{1}{2}(y - 1) = x^2$ is twice the change in the y-value of $y = x^2$. This shows a *vertical stretch* of 2.

As a mapping rule, this can be shown as $(x, y) \rightarrow (x, 2y + 1)$.

As in Example 1, the quadratic function $\frac{1}{2}(y - 1) = x^2$ can be written in standard form or general form. In each case, the function is $y = 2x^2 + 1$. Notice how the value 1 in the new form of the function is the y-coordinate of the vertex. Before answering the questions below, explain how to change $\frac{1}{2}(y - 1) = x^2$ to the different forms.

- In general form, what is the value of a in $y = 2x^2 + 1$?
- In general form, what is the value of b in $y = 2x^2 + 1$?
- In general form, what is the value of c in $y = 2x^2 + 1$?
- What does the value of a in the general form of a quadratic function tell you? What does the value of c tell you?

– Note –

In Example 1, you thought about what the values of a and c in the quadratic function $y = ax^2 + bx + c$ can tell you. Is it necessary to modify your thoughts based on the information in Example 2? Think about this as you complete Example 3.

Example 3

Compare the graphs of $y = x^2$ and $3(y + 2) = (x - 3)^2$. How are they alike? How are they different? Before trying the Focus questions, work with a partner to answer these questions from each of the examples.

Solution

A table of values and graphs can help you compare the two functions.

x	–3	–2	–1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9
$3(y + 2) = (x - 3)^2$	10	$\frac{19}{3}$	$\frac{10}{3}$	1	$\frac{-2}{3}$	$\frac{-5}{3}$	–2

– Note –

Remember how to read the table of values from Examples 1 and 2. The second and third rows give the y-value of each function for the given value of x.

Think about...

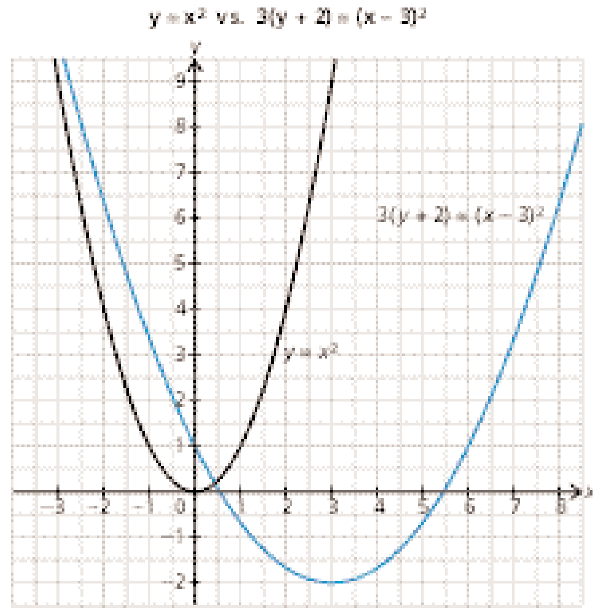
Transformations

How are the translations shown in the equation $3(y + 2) = (x - 3)^2$? How is the vertical stretch shown? Why do you think this is called the *transformational form* of the function?

Think about...

Standard Form

How can you find the vertex of the quadratic function from the standard form? How is it like finding the vertex from the transformational form? How is it different?



The vertex of the parabola of $3(y + 2) = (x - 3)^2$ is located at (3, –2), while the vertex of the parabola of $y = x^2$ is located at (0, 0). This shows that the function $3(y + 2) = (x - 3)^2$ has a vertical translation of –2 and a *horizontal translation* of 3 when compared to the graph of the function $y = x^2$. As a point on each parabola moves away from its vertex, a pattern is formed.

- $y = x^2$ — over 1 up 1, over 2 up 4, over 3 up 9
- $3(y + 2) = (x - 3)^2$ — over 1 up $\frac{1}{3}$, over 2 up $\frac{4}{3}$, over 3 up $\frac{9}{3}$ or 3

The patterns show that, for each unit increase of x from the vertex, there is a vertical stretch of $\frac{1}{3}$. As a mapping rule, this can be shown as $(x, y) \rightarrow \left(x + 3, \frac{1}{3}y - 2\right)$.

As was the case in Examples 1 and 2, the quadratic function $3(y + 2) = (x - 3)^2$ can be written in standard form or general form. In standard form, the function is $y = \frac{1}{3}(x - 3)^2 - 2$. In general form, the function is $y = \frac{1}{3}x^2 - 2x + 1$. Before answering the questions below, explain how you could change $3(y + 2) = (x - 3)^2$ to standard form and general form.

- In general form, what is the value of a in $y = \frac{1}{3}x^2 - 2x + 1$?
- In general form, what is the value of b in $y = \frac{1}{3}x^2 - 2x + 1$?
- In general form, what is the value of c in $y = \frac{1}{3}x^2 - 2x + 1$?

- What does the value of a in the general form of a quadratic function tell you? What does the value of c tell you?

Now compare your work with those of others in the class before trying the Focus questions.

Focus Questions

1. Summarize what the values of a and c in the general form of the quadratic function $y = ax^2 + bx + c$ tell you. Compare your results with those of others in the class.

2. Work with a partner and list the advantages of writing a quadratic function in transformational form, standard form, and general form. Describe the pieces of information you get from each form. Use examples of your own to help support your list of advantages.

3. (a) Graph and compare each pair of quadratic functions and list their similarities and differences.

(i) $y = x^2$ and $y = -x^2$

(ii) $\frac{1}{2}y = x^2$ and $-\frac{1}{2}y = x^2$

(iii) $\frac{1}{3}y = x^2$ and $-\frac{1}{3}y = x^2$

(iv) $\frac{3}{4}y = x^2$ and $-\frac{3}{4}y = x^2$

(b) Why is one graph in each pair called a *reflection* of the other graph?

(c) Explain why a reflection about the x-axis changes the sign but not the magnitude of y-values of a graph.

(d) Compare the vertical stretch of each pair of graphs in part (a). What do you notice?

4. State the location of the vertex of the parabola of each quadratic function.

(a) $\frac{1}{4}y = x^2$

(b) $y = 4x^2 + 16$

(c) $\frac{1}{2}(y - 2) = (x - 3)^2$

(d) $y - 5 = (x - 4)^2$

(e) $y = -2(x - 3)^2 - 2$

(f) $\frac{1}{4}(y - 1) = (x - 2)^2$

5. Use the functions in Question 4.

(a) Write a mapping rule to show the transformation of the graph of $y = x^2$ onto the graph of each function.

(b) State the vertical stretch and the y-intercept of each function. Work with a partner and explain how you found each.

(c) Write each function in general form and compare the general form to the mapping rule. How are the transformational and general forms alike? How are they different? What can each tell you about the function?

Check Your Understanding

6. For each of the following functions:

- Write a mapping rule to describe how to map the function $y = x^2$ onto the new function.
- Write the new function in general form.
- Write the y-intercept and the vertical stretch factor for each.

(a) $y + 2 = x^2$

(b) $3(y + 4) = (x - 2)^2$

(c) $y = 2(x + 1)^2 + 1$

(d) $y - 1 = x^2$

(e) $y = (x + 1)^2 + 1$

(f) $\frac{1}{2}(y - 1) = (x + 1)^2$

(g) $-\frac{1}{2}(y + 1) = (x + 1)^2$

7. Look at the functions in Question 6. Which form of each function—transformational form, standard form, or general form—was easiest to use when finding the vertical stretch factor? the y-intercept? Give reasons for your answer.

8. Describe what happens to the shape and vertical stretch of the graph of $\frac{1}{k}y = x^2$ or $y = kx^2$

(a) as k increases from 1 to 1000

(b) as k decreases from 1 to close to 0

(c) as k decreases from 0 to –1000

9. Sketch the graph of the image of $y = x^2$ under each mapping rule. Write the function in both general form and transformational form. Which form did you write first? Why?

(a) $(x, y) \to (x - 2, y)$

(b) $(x, y) \to (x - 5, y - 3)$

(c) $(x, y) \to (x + 4, 2y)$

(d) $(x, y) \to (x - 1, -\frac{1}{2}y)$

Think about...

Question 8

What is the relationship between the coefficient of the x^2-term and the vertical stretch of the graph of a quadratic function written in general form?

– Note –

In Question 8, the variable k can be written as a, b, or in fact any variable. Be careful when you look at other forms.

10. The function $y = -2x^2 - 4x - 1$ is written in general form. The equivalent transformational form is written $-\frac{1}{2}(y - 1) = (x + 1)^2$.

(a) Write the equation of the function in standard form.

(b) Use either of the forms to list some properties of the graph of the quadratic function.

(c) Which form provides more information about the parabola?

11. Each vertex listed below is a horizontal or vertical translation of the vertex (0, 0) of the parabola $y = x^2$.

- Create a quadratic function whose graph has a vertex at each location. Write each function in transformational form, standard form, and general form.
- Which form was easiest to create first?
- Is there more than one function that has a vertex at each location? Explain.

(a) (0, 8) (b) $\left(0, \frac{3}{4}\right)$

(c) (0, –4.5) (d) (2, 0)

(e) (–2, 2)

12. Prove that a vertical translation of q units upward of the graph of the basic quadratic function, followed by a reflection about the x-axis, produces a different quadratic function than a reflection about the x-axis of the graph of the basic quadratic function followed by a vertical translation upward of q units.

13. Create two different mapping rules that move the vertex of the graph of the basic quadratic function from (0, 0) to (5, 5). Write each function in both transformational form and general form. Which function did you create first? Why?

CHALLENGE yourself

The basic quadratic function $y = x^2$ is translated vertically q units, where q is any real number.

(a) Find the points of intersection with the graph of the function $y = -x^2$.

(b) What linear function can be drawn through both points of intersection found in part (a)?

FOCUS D Creating the Transformational Form of a Quadratic

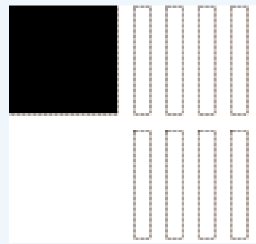
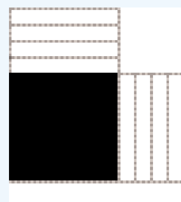
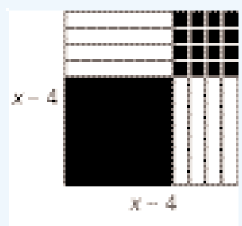
Example 4

The path of a golf ball is a parabola described by the function $y = -2x^2 + 16x$, where x represents the time in seconds after the ball was hit and y represents the ball's height in metres. What is the maximum height of the ball and when was it reached?

Solution

In order to answer this question, you need to know the coordinates of the vertex of the parabola. You can find the coordinates of the vertex by writing the quadratic function in transformational form.

One way to write the function in transformational form is by completing the square, as shown using the algebra tiles below.

y	$= \quad -2x^2 + 16x$	
$-\frac{1}{2}y$	$x^2 - 8x$	Step 1
$-\frac{1}{2}y$		Step 2
$-\frac{1}{2}y$		Step 3
$-\frac{1}{2}y + 16$		Step 4
$-\frac{1}{2}y + 16$	$(x - 4)^2$	Step 5

The function $y = -2x^2 + 16x$ can be expressed in transformational form as $-\frac{1}{2}(y - 32) = (x - 4)^2$.

Focus Questions

14. Describe each step of Focus D for rewriting the general form of the quadratic function in transformational form.

15. Complete the square on each of the following.

(a) $y = x^2 + 6x$ (b) $y = x^2 - 4x$ (c) $y = x^2 + 5x$

(d) $y = x^2 - 6x$ (e) $y = x^2 + 7x$ (f) $y = x^2 - 3x$

What appears to be the relationship between the coefficient b and the constant added to complete the square?

Think about...

The Expression

Why is the expression $-\frac{1}{2}y + 16$ equivalent to the expression $-\frac{1}{2}(y - 32)$?

16. Why is the expression $(x - 4)^2$ called a perfect square?

17. Use completing the square to create the transformational form of each quadratic function.

(a) $y = x^2 + 4x$ (b) $y = x^2 - 5x + 1$ (c) $y = x^2 - 12x$

(d) $y = x^2 + 2x + 6$ (e) $y = -x^2 - 9x$ (f) $y = x^2 + \frac{3}{4}$

(g) $y = x^2 - \frac{1}{3}x$

18. (a) Explain how the general form $y = -2x^2 + 16x$ can be rewritten algebraically to find the maximum height of the golf ball and when it occurred.

(b) Graph the general form to confirm your answer to part (a).

19. Inside a golf dome, a golf ball hit from a 1-m platform follows a parabolic path described by the function $y = -6x^2 + 12x + 1$, where x represents the time in seconds after the ball is hit and y represents the ball's height in metres from the floor. The height of the golf dome from floor to ceiling is 10 m.

(a) Will the ball strike the ceiling?

(b) For how many seconds will the ball travel in the air?

20. Use the coefficient a to factor each quadratic expression. Create the transformational form of each quadratic function.

(a) $y = 4x^2 + 8x$ (b) $y = 4x^2 + 4x$ (c) $y = 9x^2 + 6x$

(d) $y = 4x^2 - 4x$ (e) $y = -4x^2 - 12x$ (f) $y = 3x^2 - 2x$

21. Explain how the transformational form of a quadratic function such as those in Questions 17 and 20 can be used to find the

(a) location of the vertex

(b) equation of the line of symmetry

(c) vertical stretch

(d) horizontal and vertical translation of the basic quadratic function $y = x^2$

22. A tossed stone follows a parabolic path described by the function $y = -2x^2 + 10x$, where x is the time in seconds after the stone is tossed and y is the stone's height in metres.

(a) For how many seconds is the stone in the air?

(b) What is the maximum height of the stone?

(c) How high is the stone after 3 s?

(d) How long does it take the stone to reach a height of 12 m?

CHALLENGE yourself

Show that any quadratic expression of the form $ax^2 + bx$ can be written as a perfect square after adding the constant $\frac{b^2}{4a}$, if $a \neq 0$.

Check Your Understanding

23. Rewrite each equation of the function in either standard form, transformational form, or general form.

(a) $y = x^2 - 2x$

(b) $y = -4x^2 + 12x + 12$

(c) $-(y - 6) = x^2$

(d) $y = \frac{1}{2}x^2 + 12x - 3$

(e) $y = 5(x - 4)^2 + 3$

(f) $2(y - 3) = 3(x - 4)^2$

24. Write a quadratic function of the form $y = ax^2 + bx + c$, for which a, b, and c are not equal to zero. Exchange functions with another student.

(a) Express the quadratic function you were given in its transformational form.

(b) Use it to find the vertex and vertical stretch.

25. A parabola has its vertex at (3, 10). What additional information is needed to determine the transformational form of the quadratic function?

26. A submarine travelling in a parabolic arc ascends to the surface. The path of the submarine is described by $y = 2x^2 - 10x - 50$, where x represents the time in minutes and y represents the submarine's depth in metres.

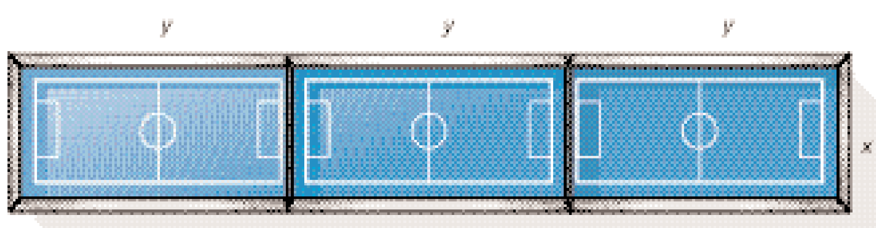
(a) How deep is the submarine initially?

(b) For how long is the submarine underwater?

(c) What maximum depth does it reach?

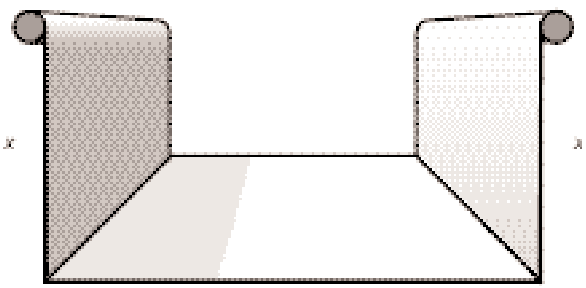
27. An environmental group creates a rectangular habitat with a lake forming one side of the habitat. If they use 48 km of fencing, what dimensions ensure a maximum area of the habitat? What is the maximum area of the enclosed habitat in square kilometres?

28. Fencing is used to divide a rectangular playing field into three congruent sections as shown. If the total length of fencing is 800 m, what are the values of x and y that maximize the area?



29. Find two numbers whose difference is 13 and whose squares when added together yield a minimum.

30. An eavestrough is made by bending a rectangular strip of metal 30 cm wide.



The carrying capacity of the eavestrough is maximized when the cross-sectional area is maximized. For what value of x will the area and carrying capacity be maximized?

31. (a) An ice cream specialty shop currently sells 240 ice cream cones per day at a price of \$3.50 each. The shop finds that sales will increase by 60 cones daily for each \$0.05 decrease in price. If the shop pays \$2.00 for each ice cream cone, what price will maximize its profit?

(b) What price will maximize the shop's profit if each decrease in price of \$0.25 results in an increase of 60 cones?

32. The right-side expression of each function is a perfect square. Assuming that $b > 0$ and $c > 0$, find b and c and write each expression as a perfect square.

(a) $y = x^2 - bx + 121$ (b) $y = x^2 + bx + 15$

(c) $y = x^2 - bx + 49$ (d) $y = 4x^2 - 4x + c$

(e) $y = 8x^2 + 16x + c$ (f) $y = \frac{1}{4}x^2 + x + c$

33. Use the results of Question 32 to determine the location of the vertex of the graph of each function.

34. Prove that $y = (x - r)^2 + (x - s)^2$ has a minimum when $x = \frac{r + s}{2}$.

CHALLENGE yourself

(a) Express the general form of the quadratic function $y = ax^2 + bx + c$ in its transformational form.

(b) Name the coordinates of the vertex of $y = ax^2 + bx + c$.

(c) Use your results to find the vertices of $y = 6x^2 + 24x - 7$ and $y = -\frac{1}{2}x^2 - 3x + 1$.

Focus E: Determining Quadratic Functions from Parabolas

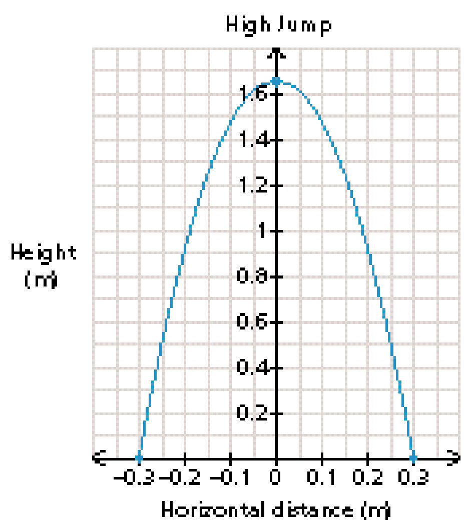
If the vertex and at least one other point of a parabola are known, the transformational form of the quadratic function can be found using two methods.

Example 5

In 1900, the standing high jump was introduced as an event in the Olympic Games. Ray Ewry of the USA won the event with a jump of 1.65 m. If he started his vertical jump at a horizontal distance of 30 cm or 0.30 m from the crossbar:

- the start can be represented as (–0.30, 0)
- the vertex can be represented as (0, 1.65), and
- the end can be represented as (0.30, 0).

The graph shows the parabolic path of his jump.



Find the transformational form of the quadratic function describing the path of the jump.

Solution

Method 1

We can determine the quadratic function by considering transformations on the basic quadratic function $y = x^2$.

The vertex (0, 1.65) represents a horizontal translation of 0 and a vertical translation of 1.65. These translations can be represented by the quadratic function $y - 1.65 = x^2$.

Compare the pattern as the x-value increases from the vertex of the high-jump function and the function $y = x^2$:

- $y = x^2$ over 0.3 up 0.09
- high-jump function over 0.3 down 1.65

Thus, the size of the vertical stretch factor is $\frac{1.65}{0.09}$ or 18.33. Because the pattern for the high jump is *down* while the pattern for $y = x^2$ is *up*, there is a reflection.

Because the jump path has a maximum point and it opens downward, it must be a reflection of the graph of $y = x^2$ about the x-axis. The quadratic function describing the path of the jump is

$-\frac{1}{18.33}(y - 1.65) = x^2$ or $-\frac{3}{55}(y - 1.65) = x^2$.

Method 2

We know that the transformational form of the quadratic function is $\frac{1}{k}(y - q) = (x - p)^2$, where k represents the vertical stretch, p represents the x-coordinate of the vertex, and q represents the y-coordinate of the vertex.

We can substitute the jump vertex data to get this equation:

$\frac{1}{k}(y - 1.65) = (x - 0)^2$

We know that (–0.30, 0) is a point on the parabola. If we substitute $x = -0.30$ and $y = 0$, the above equation becomes

$$\frac{1}{k}(0 - 1.65) = (-0.30)^2$$
$$-\frac{1.65}{k} = 0.09$$
$$k = -\frac{1.65}{0.09} = -18.33$$

The quadratic function describing the path of the jump is

$-\frac{1}{18.33}(y - 1.65) = x^2$ or $-\frac{3}{55}(y - 1.65) = x^2$.

Focus Questions

35. (a) Find the standard form of the quadratic function representing the path of the jump.

(b) Find the equation of the curve of best fit using the jump points (–0.30, 0), (0, 1.65), and (0.30, 0).

(c) How well does the equation of the curve of best fit compare with the general form of the function in part (a)?

(d) Predict the height of the jump at $x = 0.1$.

Think about...

The Location of the Vertex

What function is created if the start is represented as (0, 0), the vertex as (0.30, 1.65) and the end as (0.60, 0)?

– Note –

In Methods 1 and 2, the calculation of vertical stretch is shown using a decimal number. You could also simplify the fraction and leave it as a fraction as shown below.

$\frac{1.65}{0.09} = \frac{165}{9} = \frac{55}{3}$

Equations written in both ways are shown in each method.

Think about...

The Selection of a Point

Verify that the same quadratic function is obtained when the coordinates (0.30, 0) are substituted instead of (–0.30, 0).

Think about...

The General Form in Question 35(a)

The graph has a vertex on the y-axis. Explain how this means that the coefficient b in the general form must equal zero.